Tensor Networks at finite energy density: ETH probes

Mari-Carmen Bañuls

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J. Ignacio Cirac

arXiv:2312.00736
PRB 106, 024307 (2022)
PRX Quantum 2, 020321 (2021)
PRL124, 100602 (2020)

MAX PLANCK INSTITUTE
OF QUANTUM OPTICS

MCQST

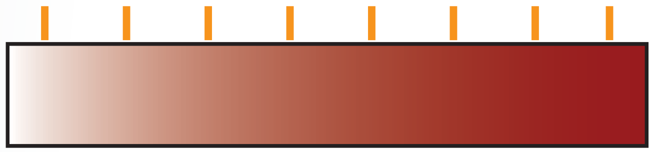
DFG FOR 5522

DFG TRR 360

30.-31.1.2025

TNS: entanglement-based ansatzes for quantum many-body states

arbitrary many-body state



d^N

$$|\Psi\rangle = \sum_{i_j} c_{i_1 \ldots i_N} |i_1 \ldots i_N\rangle$$

non-technical review: arXiv:2205.10345 (Annu Rev. CMP 2023)

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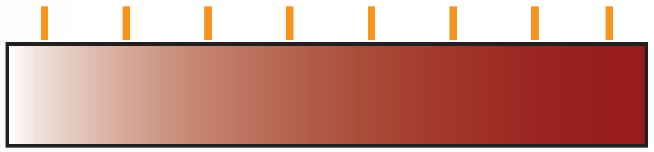

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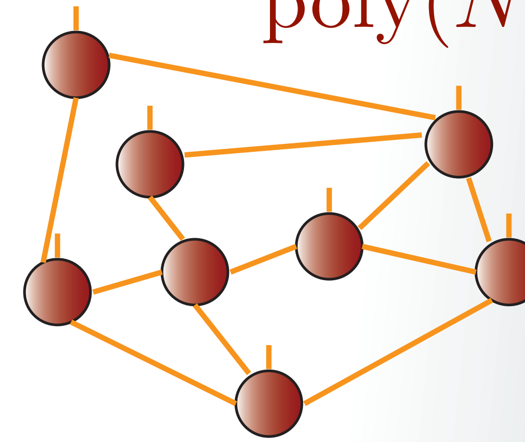
TNS: restricted family

$\text{poly}(N)$



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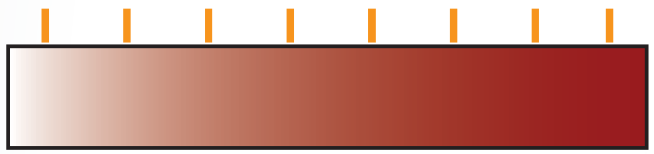

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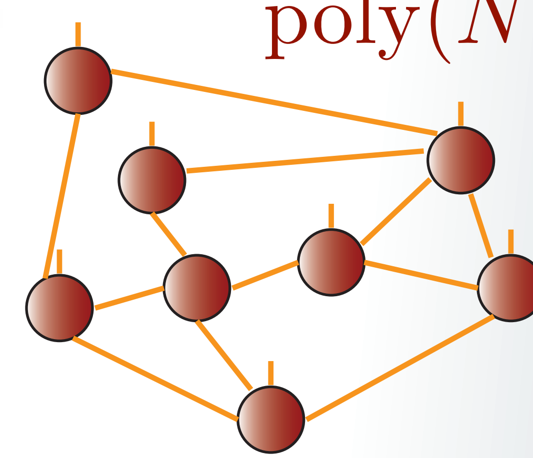

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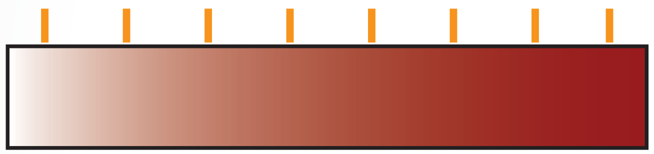

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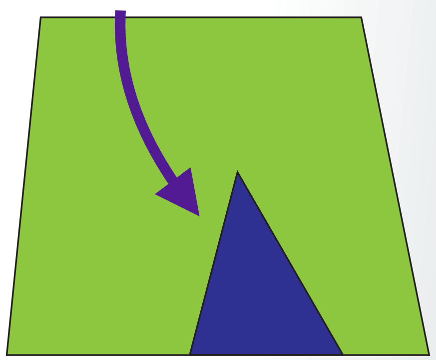
for the relevant corner of the Hilbert space



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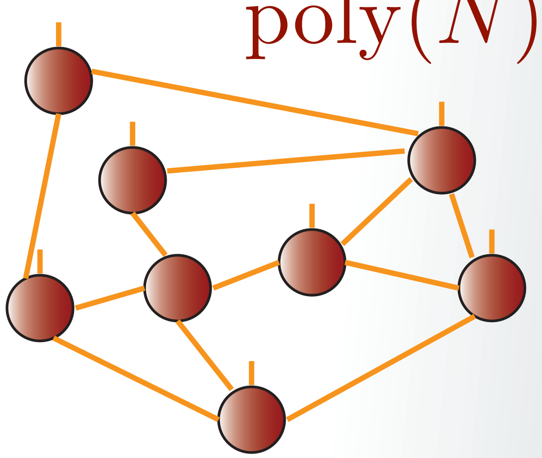

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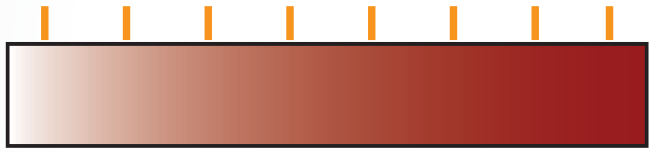

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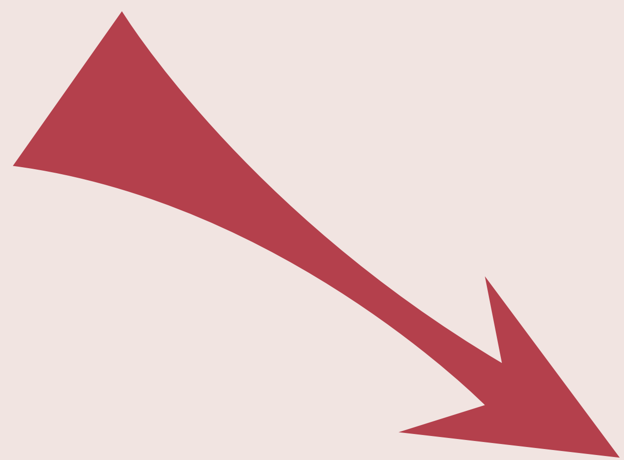


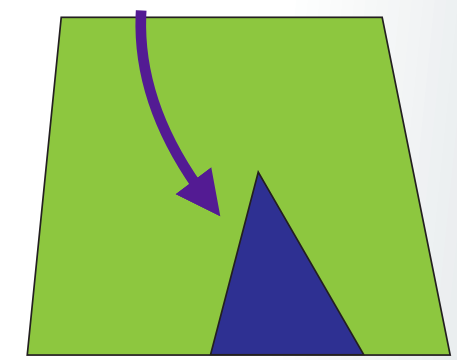
polynomial

TNS: restricted family

good ansatz for ground states and thermal equilibrium: area law

efficient numerics for large systems

for the relevant corner of the Hilbert space



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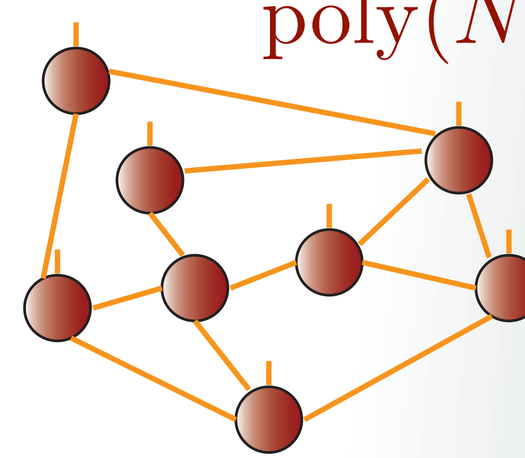

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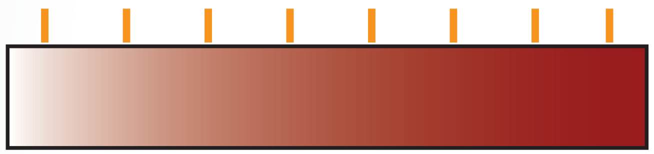

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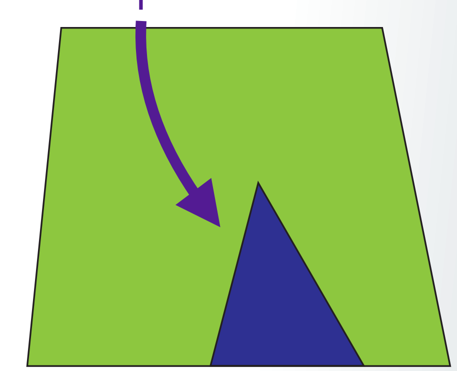
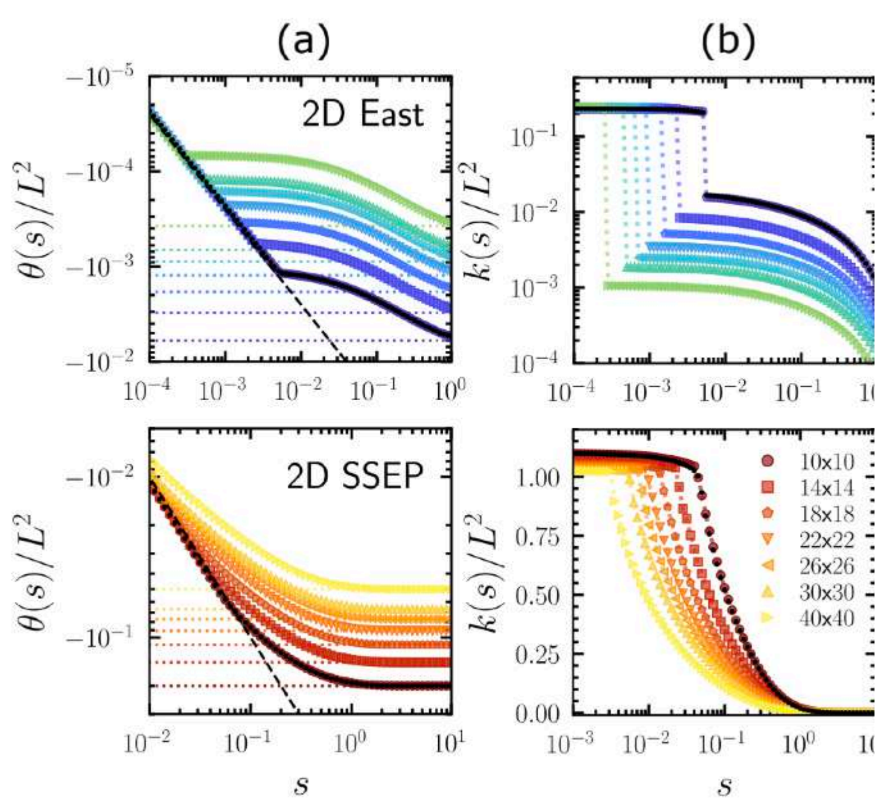


kinetically constrained models: classical stochastic dynamics

problem mapped to quantum GS

MCB, J.P. Garrahan, PRL 123, 200601 (2019),
L. Causer, MCB, J.P. Garrahan, PRL130, 147401 (2023)



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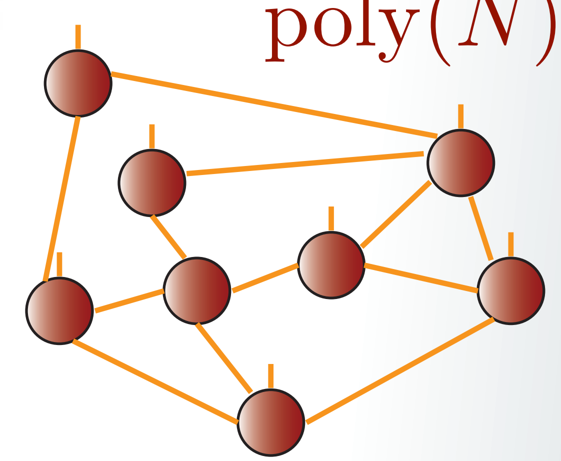

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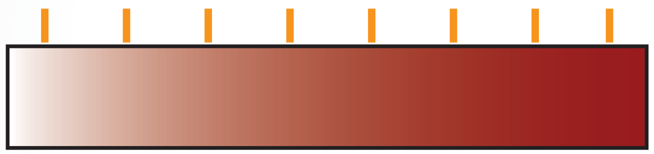

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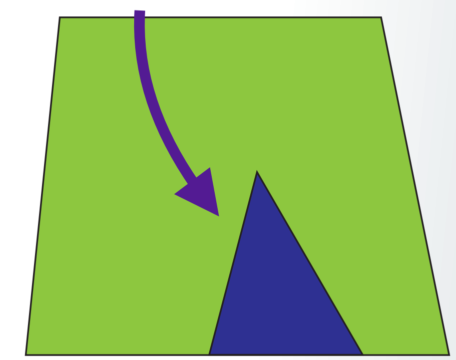

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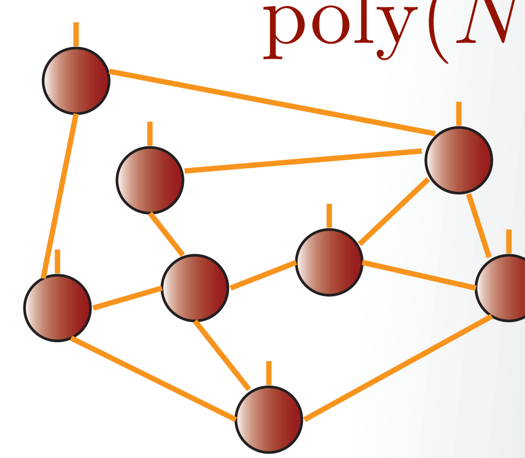
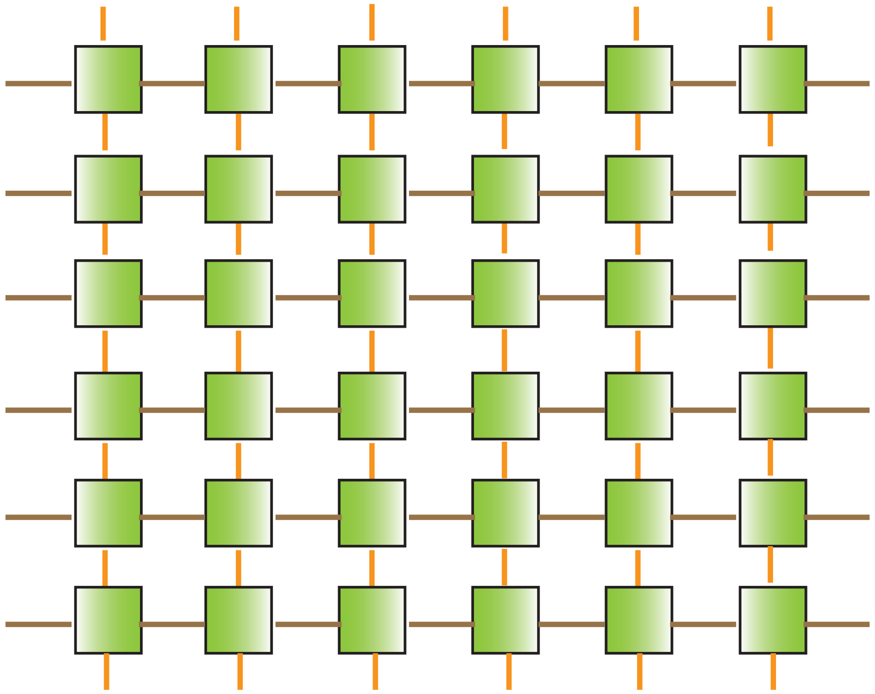


challenge: new tools for dynamical regimes

non-technical review: arXiv:2205.10345 (Annu Rev. CMP 2023)

tensor networks may describe partition functions (observables)

need to contract a TN



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Levin & Wen PRL 2008
Xie et al PRL2009; Zhao et al PRB 2010

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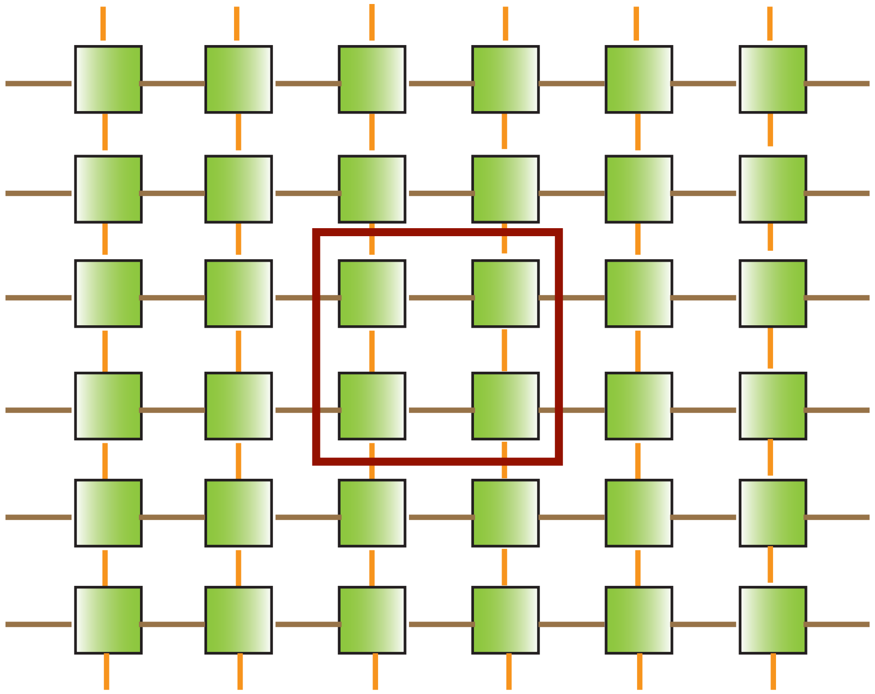

TRG approaches

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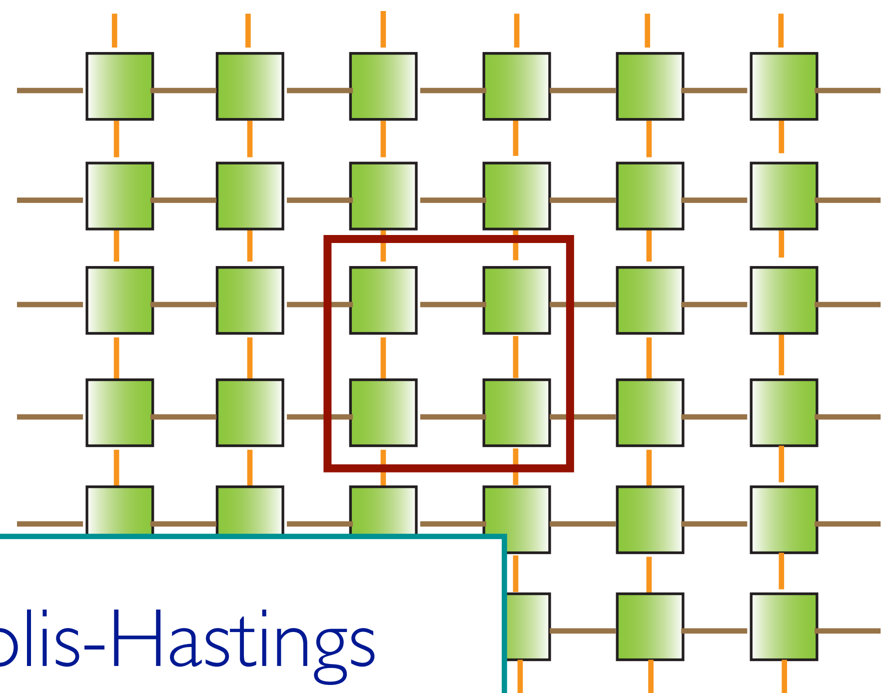
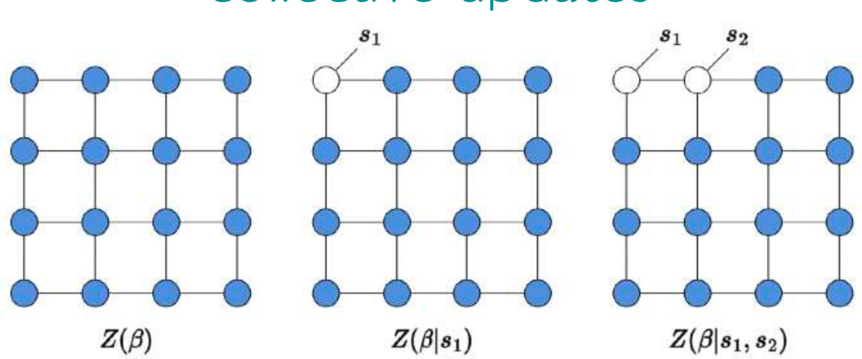


TRG approaches

TN assisted Metropolis-Hastings

collective updates



Frías-Pérez, Marien, Pérez-García, MCB, Iblisdir,
SciPost Phys. 14, 123 (2023)

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a question we want to address…

how do quantum systems thermalize?

Thermalization of quantum systems

quantum system

H $\quad |\Psi\rangle$

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$$|\Psi\rangle \to U(t)|\Psi\rangle = e^{-iHt}|\Psi\rangle$$ remains pure

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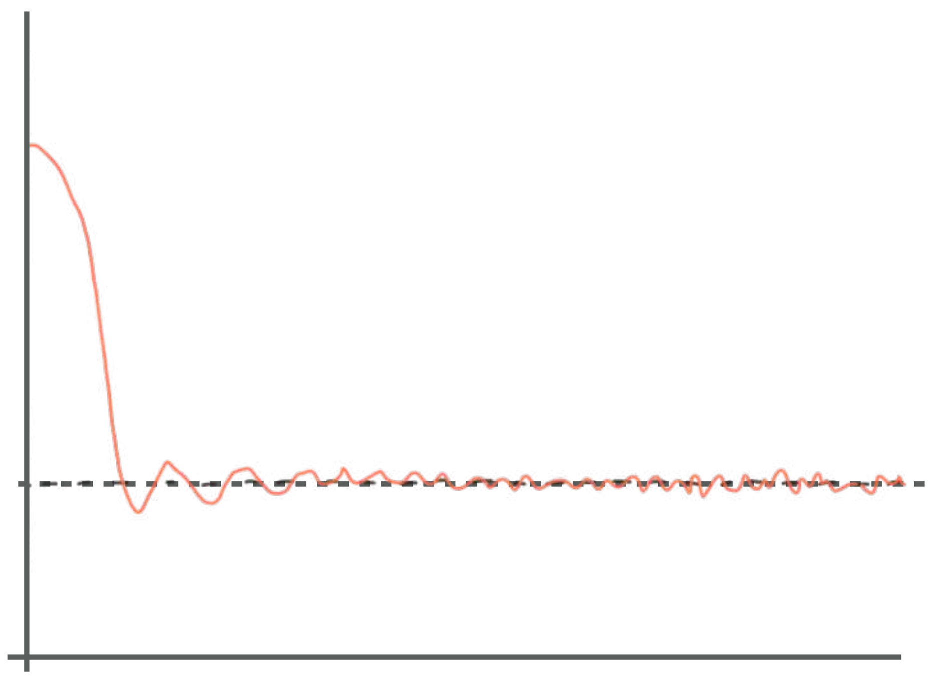

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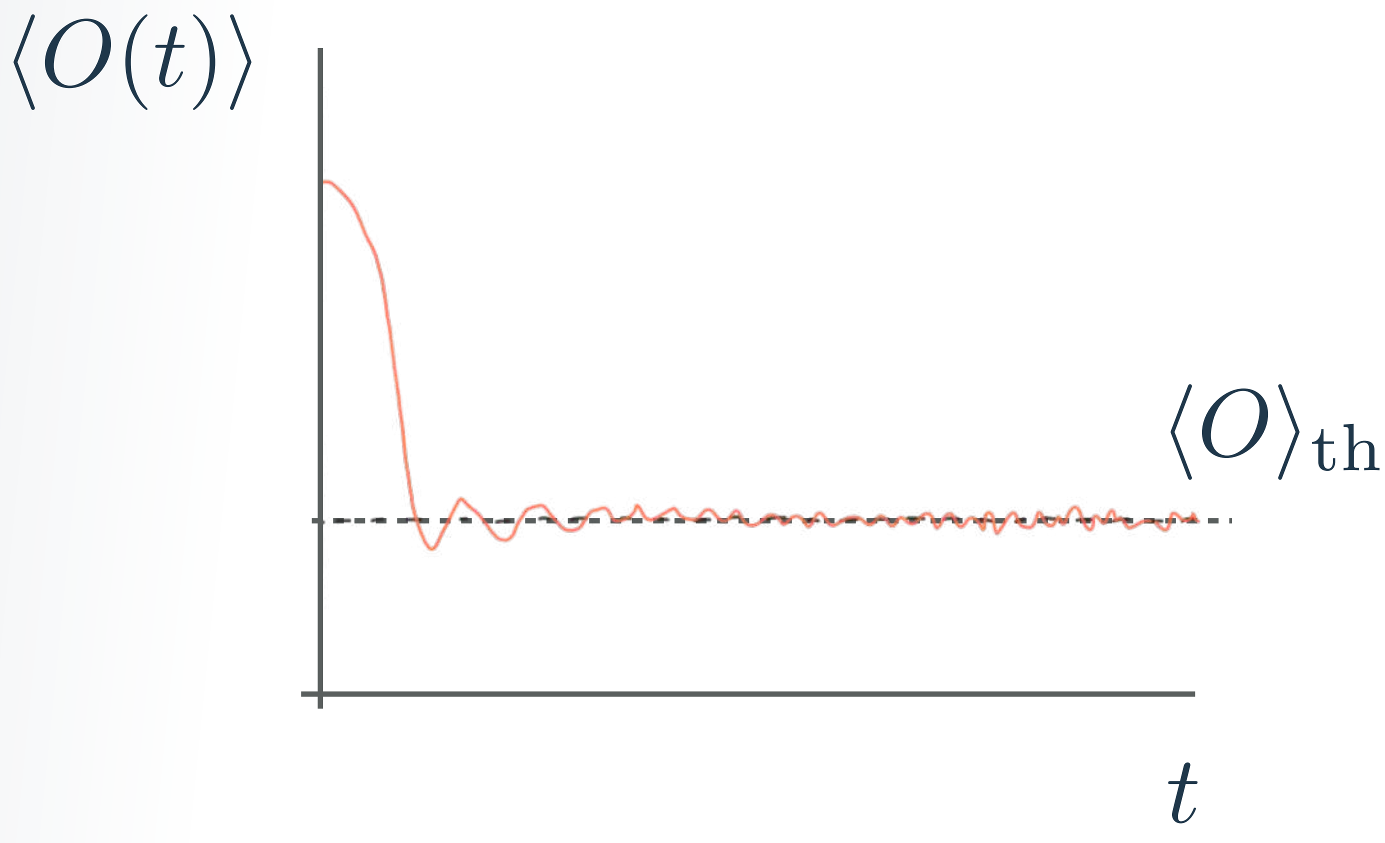

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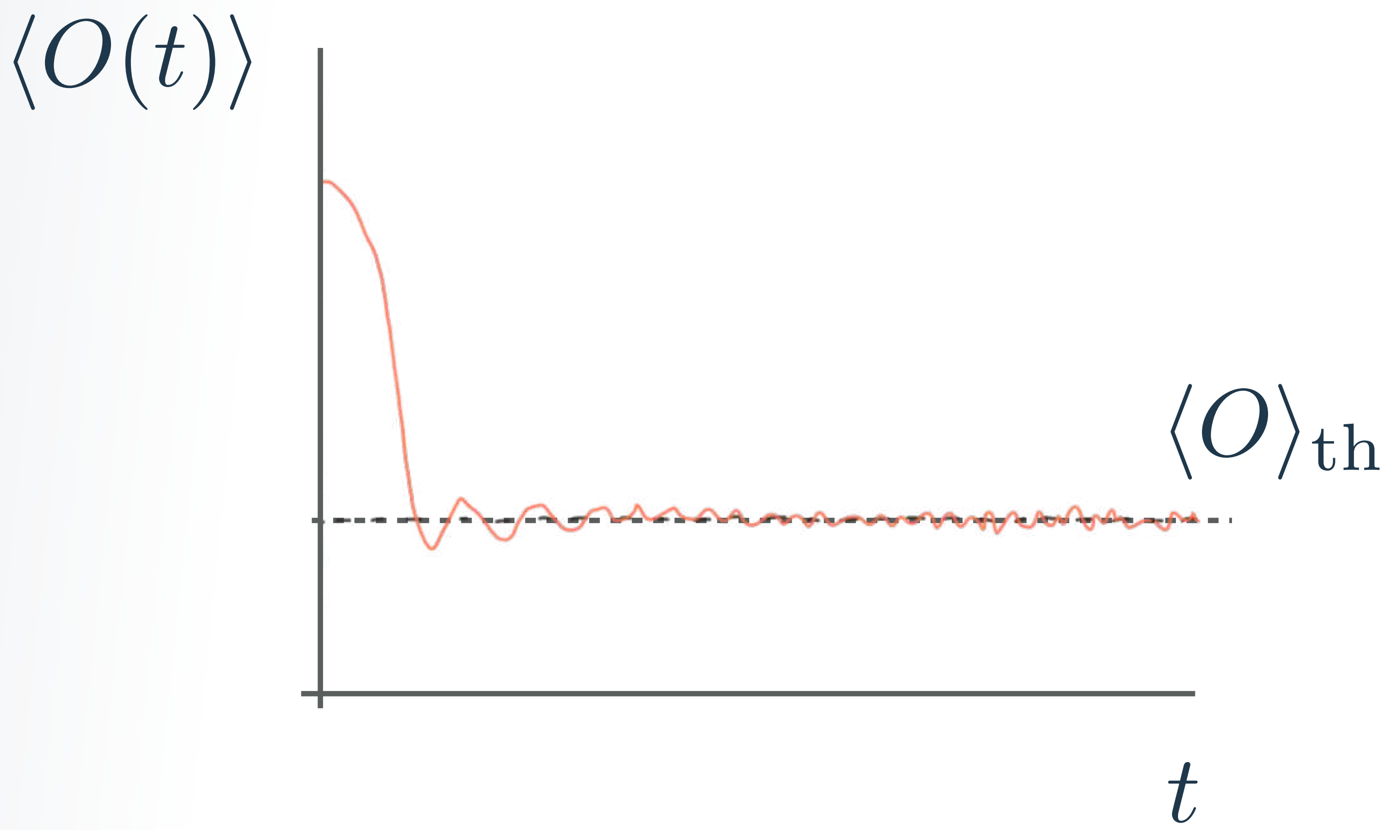

reach (remain close to) equilibrium value

predicted by thermodynamic ensemble (microcanonical)

Eigenstate Thermalization Hypothesis (ETH)

theoretical framework for quantum thermalization

review: D'Alessio et al, Adv Phys 65 (2016)

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ansatz for matrix elements of observables in energy eigenbasis

Srednicki, Deutsch 90s

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-\frac{S(\bar{E})}{2}} f_O(\bar{E},\omega)R_{mn}$$

$$\bar{E} = \frac{E_m + E_n}{2}$$

$$\omega = E_m - E_n$$

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microcanonical expectation

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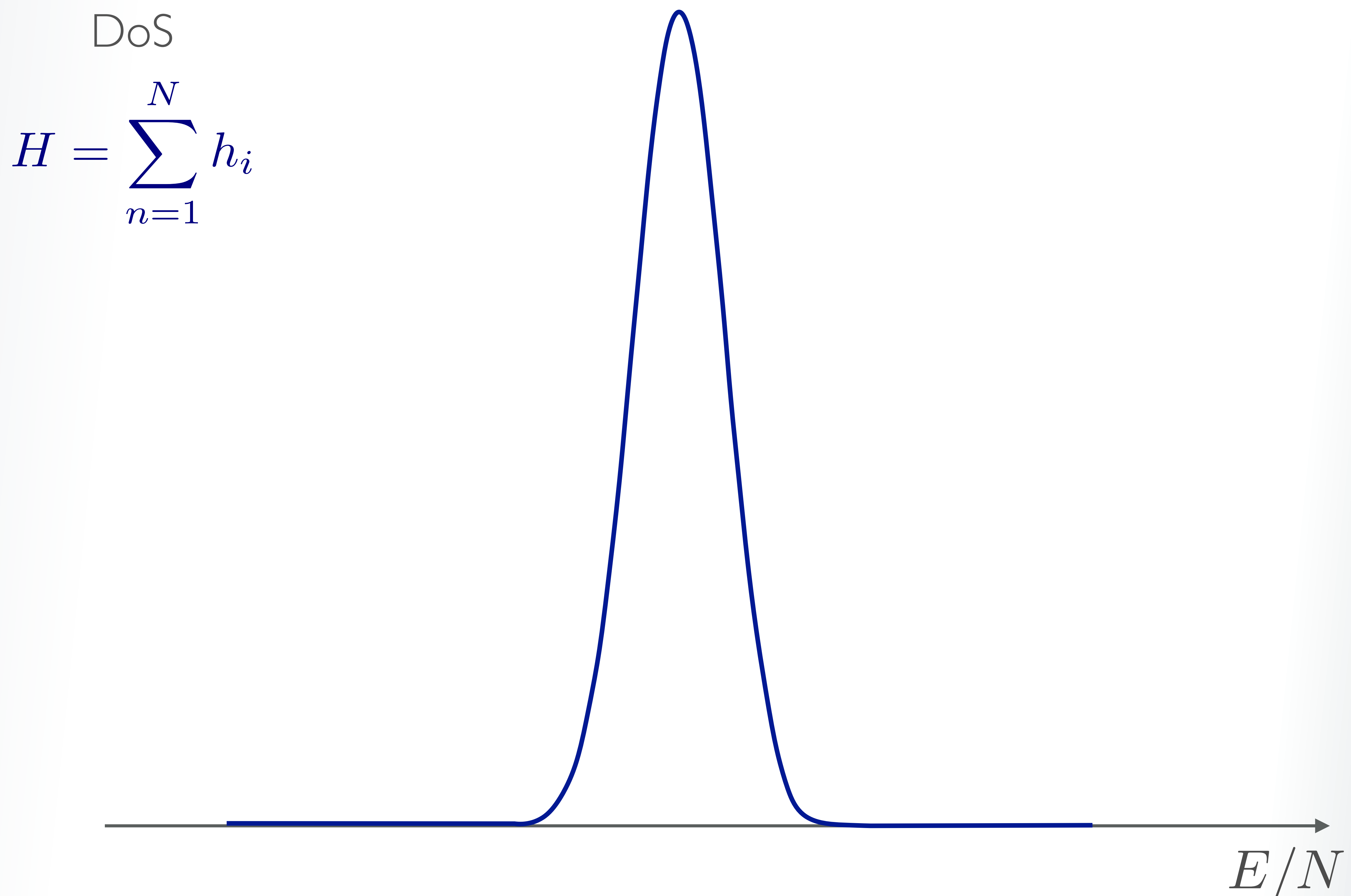
problem: exponentially large in N ⇒ Tensor Networks may be of help

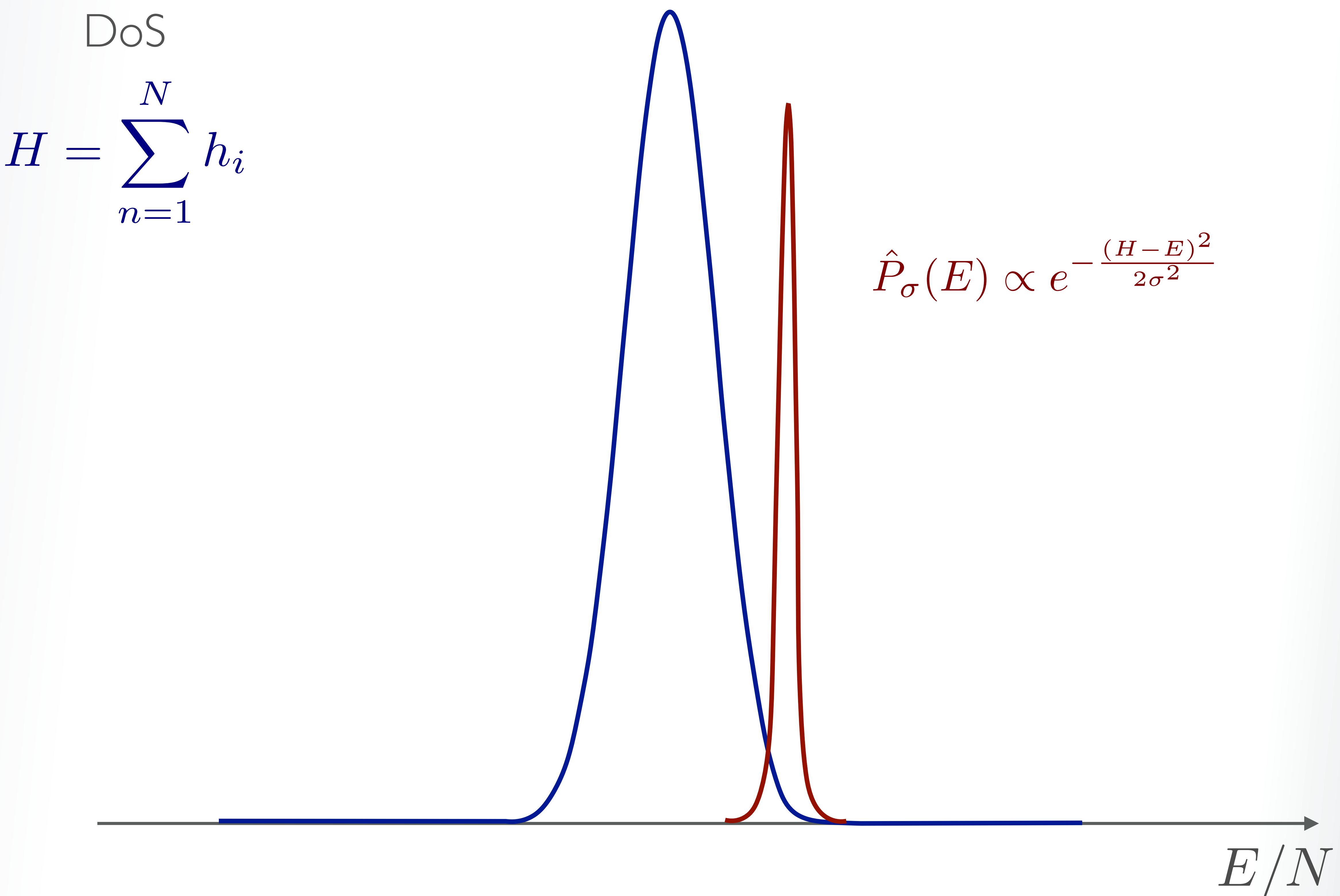
spectral (finite energy density) properties of the QMB Hamiltonian

MCB, Huse, Cirac, PRB 101, 144305 (2020)
Yang, Iblisdir, Cirac, MCB, PRL 124, 100602 (2020)
Papaefstathiou, Robaina, Cirac, MCB, PRD104, 014514 (2021)
Çakan, Cirac, MCB, PRB 103, 115113 (2021)
Lu, MCB, Cirac, PRX Quantum 2, 02032 (2021)
Yang, Cirac, MCB, PRB 106, 024307 (2022)
Luo, Trivedi, MCB, Cirac, PRB 109, 134304 (2024)

TNS are good at approximating the edges



idea: we can explore spectral (finite energy) properties by **energy filtering**



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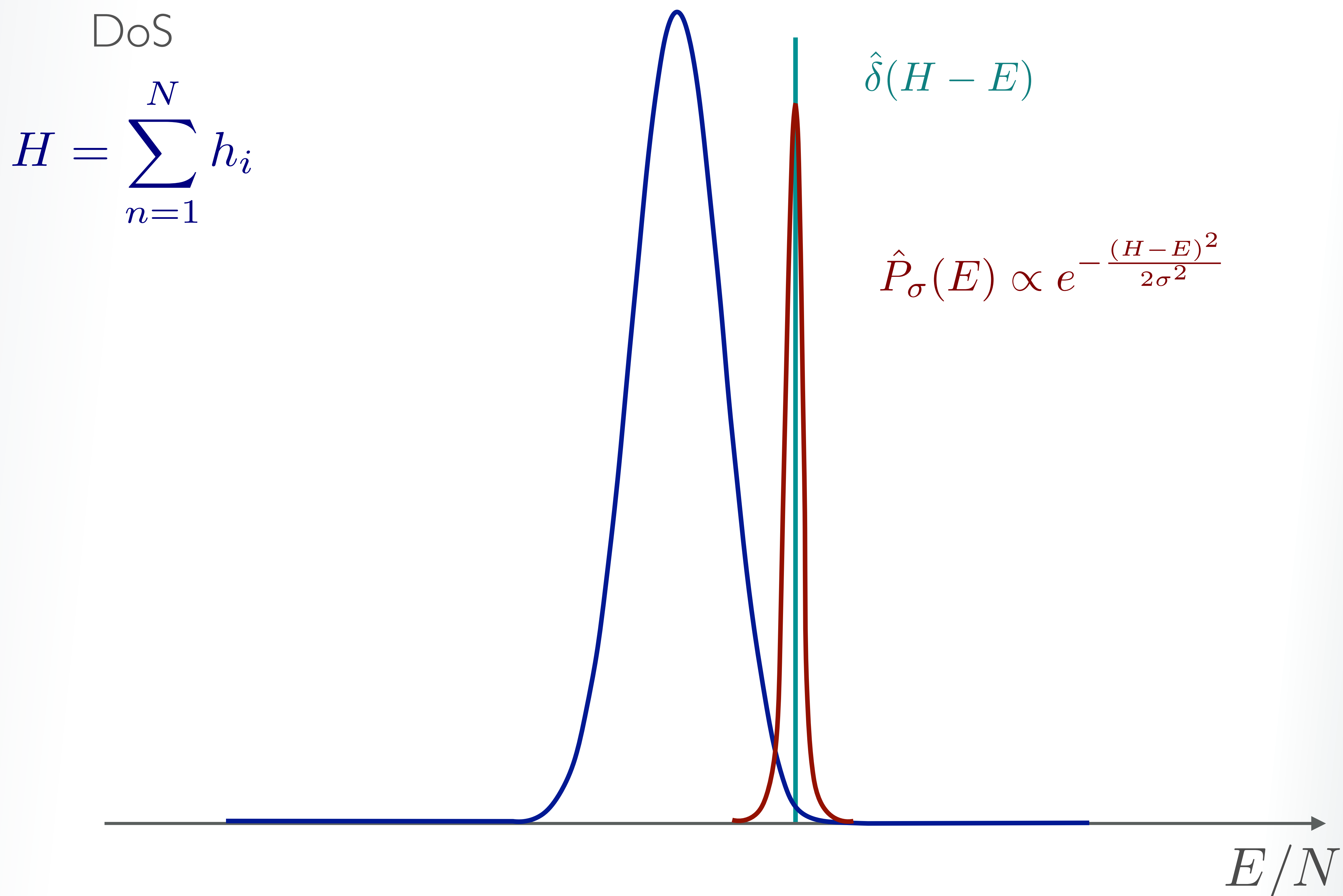
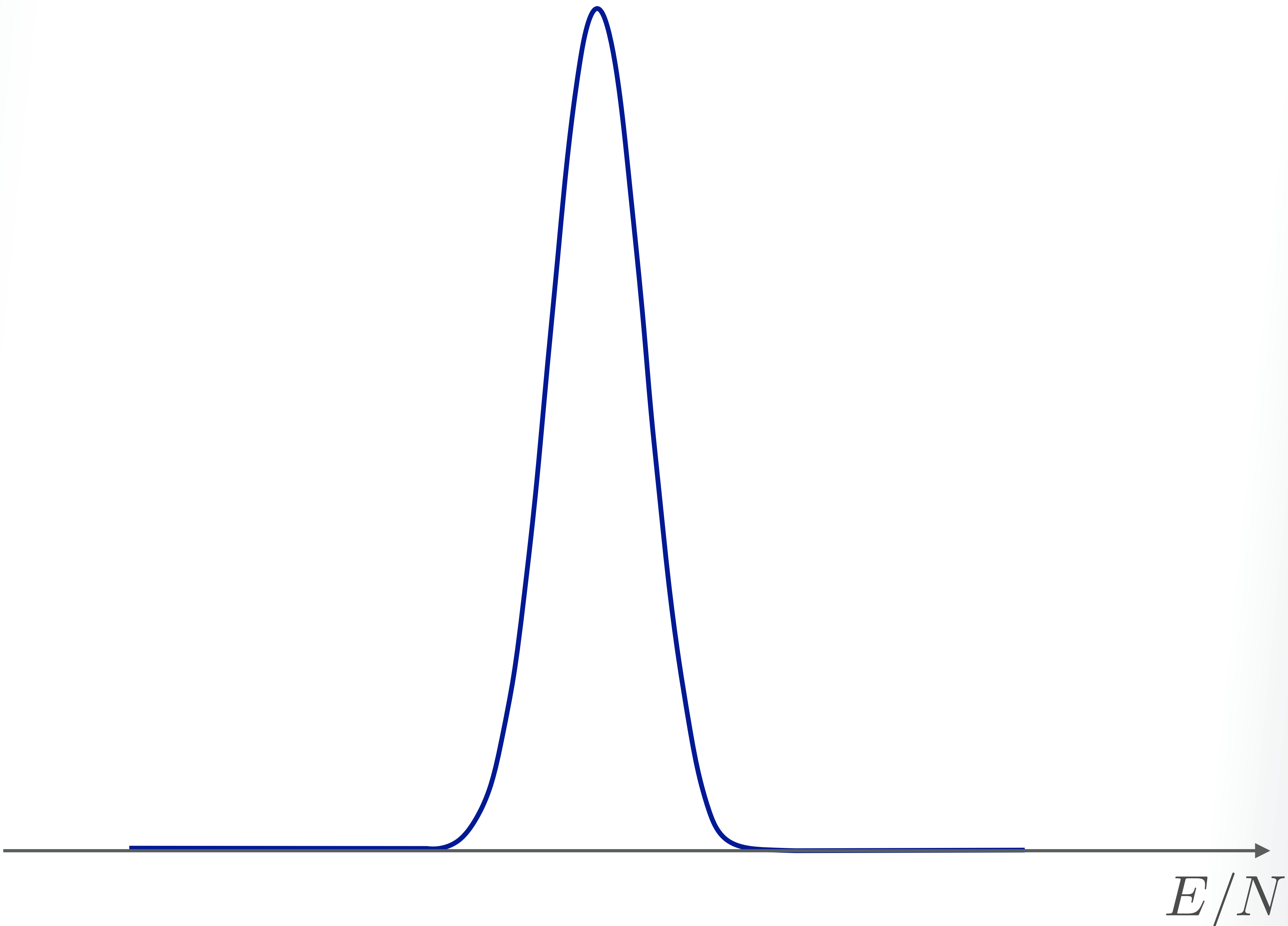


filtering a state costs entanglement



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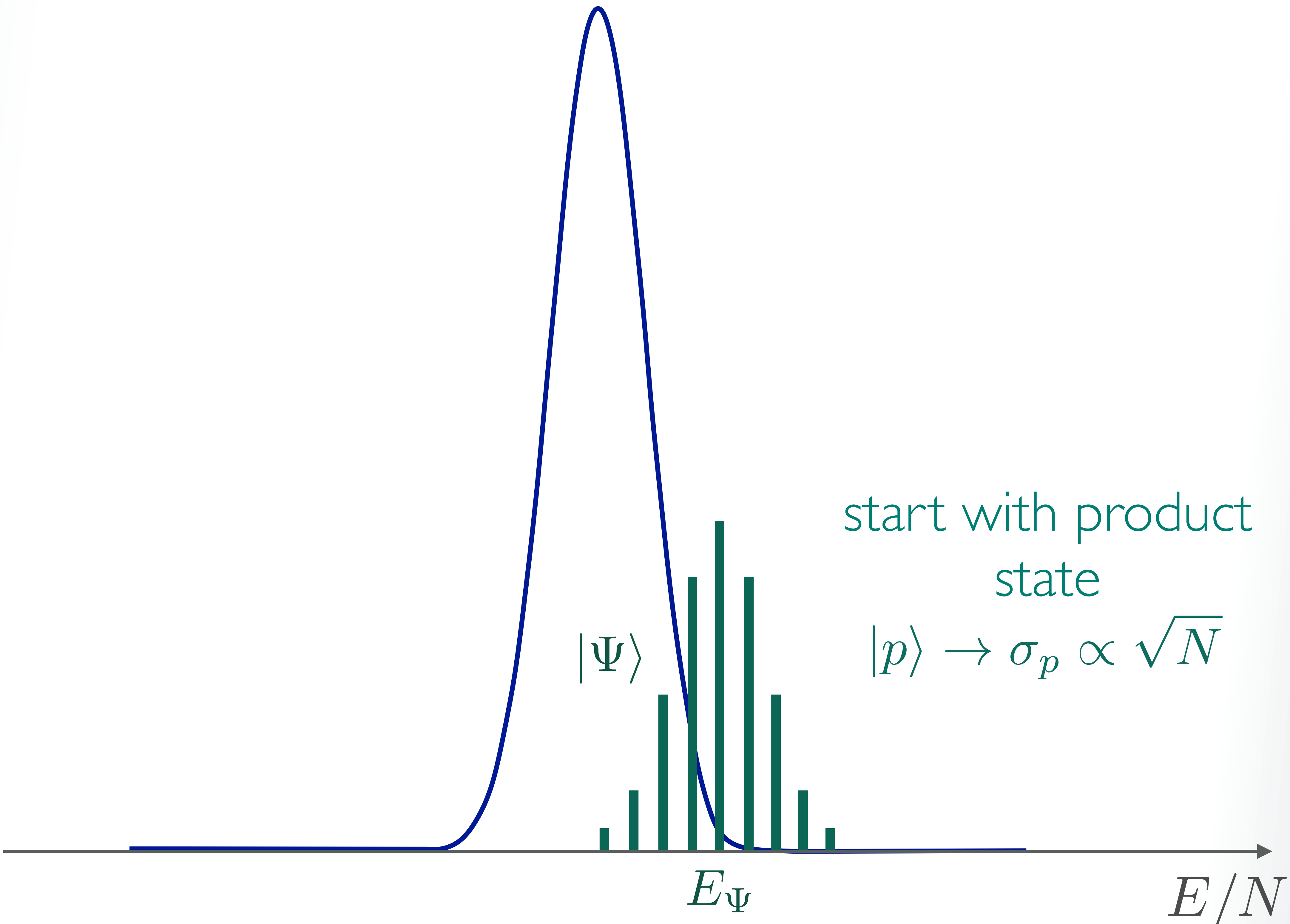

start with product state

$|p\rangle \to \sigma_p \propto \sqrt{N}$

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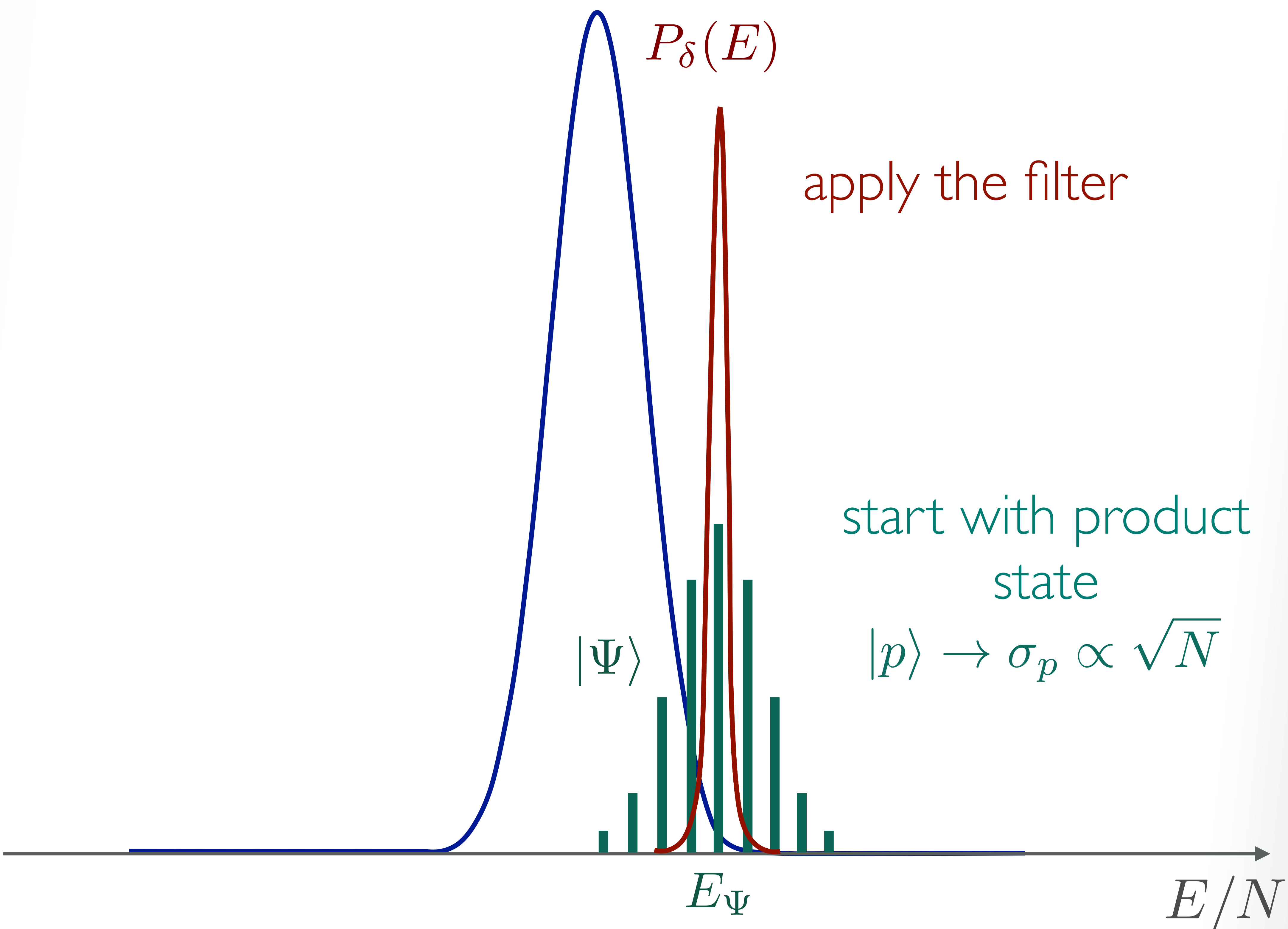


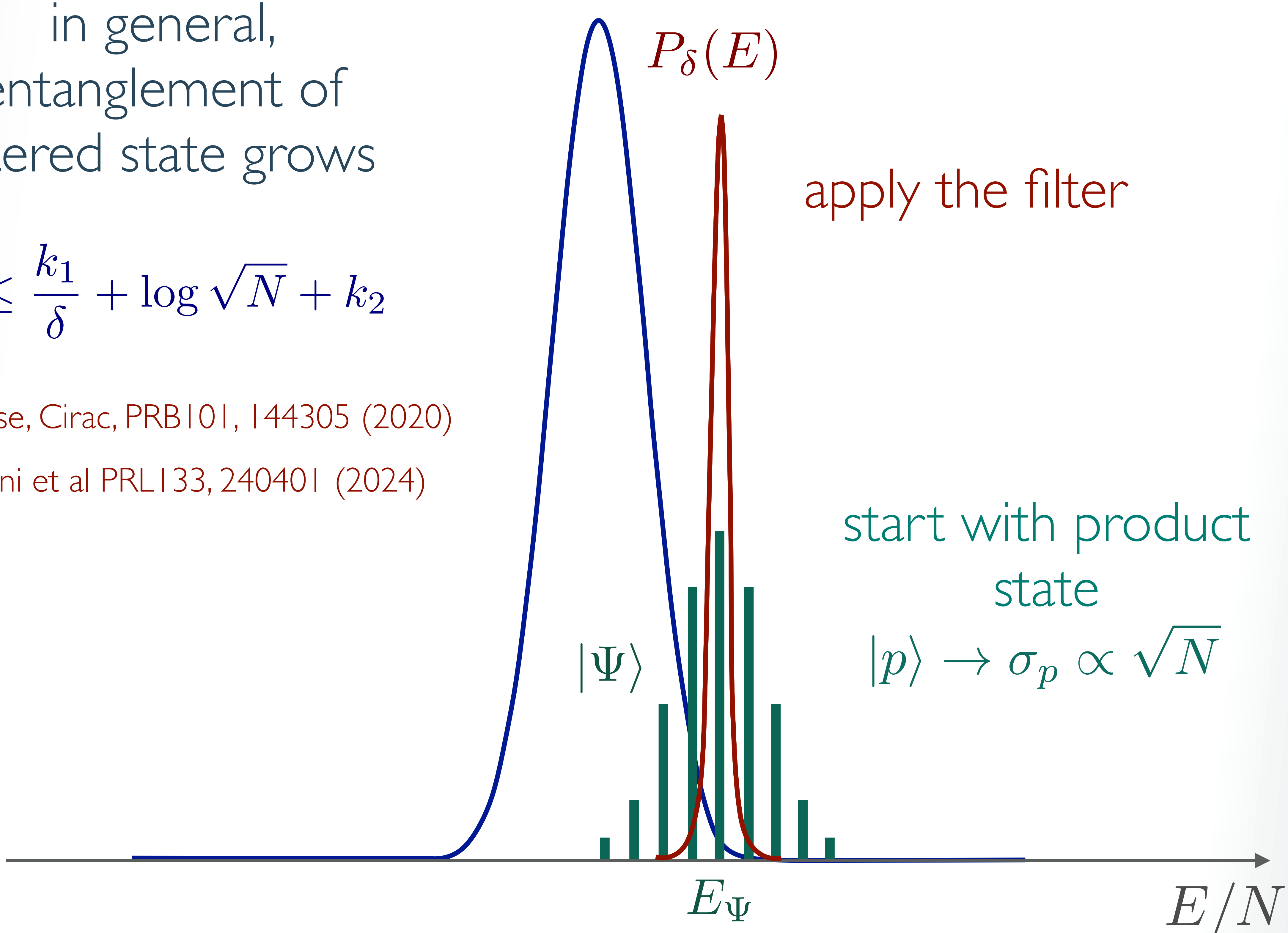
filtering a state costs entanglement

in general, entanglement of filtered state grows

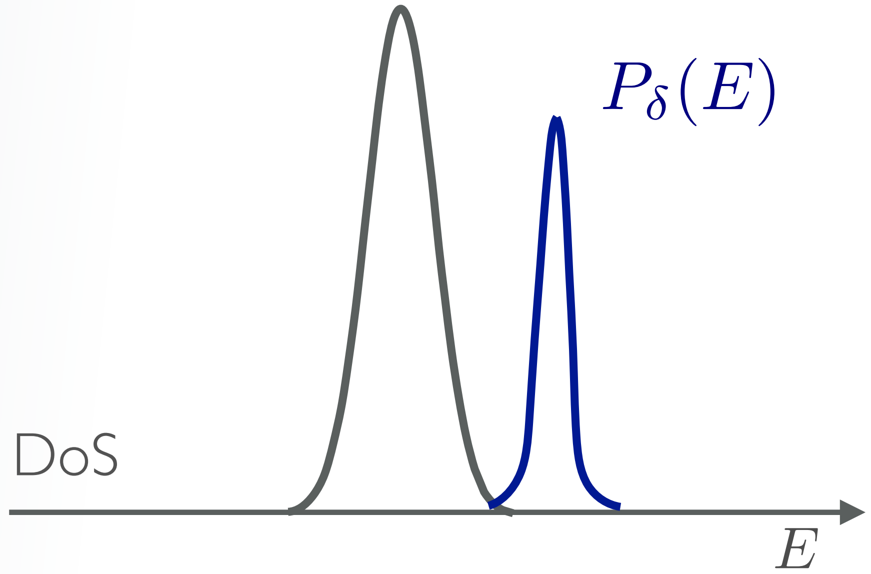
$$S \leq \frac{k_1}{\delta} + \log \sqrt{N} + k_2$$

MCB, Huse, Cirac, PRB101, 144305 (2020)

Morettini et al PRL133, 240401 (2024)



energy filter as ensemble



Lu et al. PRX Q2, 02032 (2021); Yang et al PRB 106, 024307 (2022)

energy filter as ensemble

diagonal in energy eigenbasis ⇒ microcanonical

$$\frac{\mathrm{tr}\left(O P_\delta(E)\right)}{\mathrm{tr} P_\delta(E)} \Rightarrow O(E)$$

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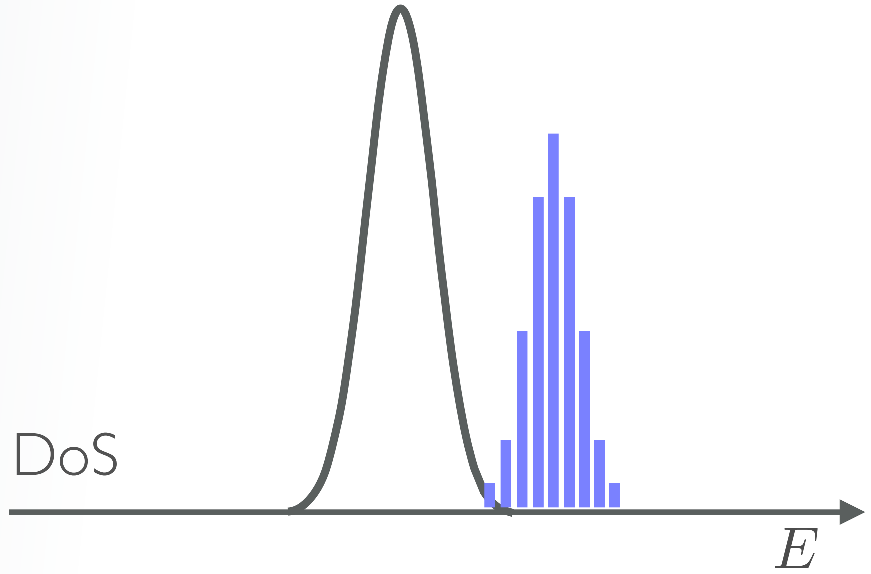

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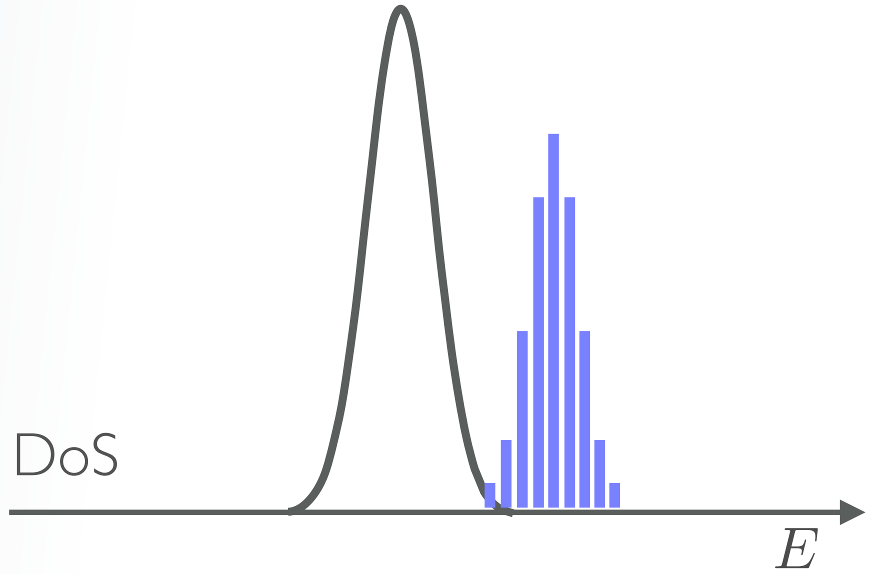

equivalent to diagonal ensemble of a certain pure state

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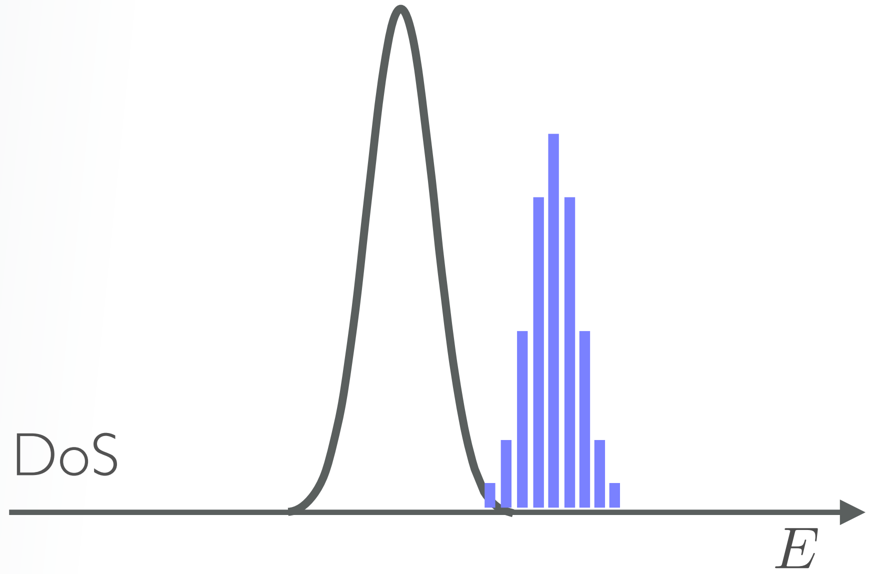

reached only after long time evolution

Lu et al. PRX Q2, 02032 (2021); Yang et al PRB 106, 024307 (2022)

implementing the filter

Gaussian filter ⇒ approximated by series of evolutions

$$\exp\left[-\frac{(H-E)^2}{2\delta^2}\right] \approx \sum_{m=-R}^{R} c_m e^{-i2mE/\alpha} e^{i2mH/\alpha}$$

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quantum inspired classical method

Lu et al. PRX Q2, 02032 (2021); Yang et al PRB 106, 024307 (2022)

algorithm

expectation values in filter ensemble

$$A_\delta(E) = \frac{\mathrm{tr}[AP_\delta(E)]}{\mathrm{tr}[P_\delta(E)]}$$

Lu et al. PRX Q2, 02032 (2021); Yang et al PRB 106, 024307 (2022)

algorithm

can be computed using Monte Carlo sampling

$$A_\delta(E) = \frac{\sum_\Psi \langle \Psi | A P_\delta(E) | \Psi \rangle}{\sum_{\Psi'} \langle \Psi' | P_\delta(E) | \Psi' \rangle}$$

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efficiently computed by quantum simulator

importance sampling (classical)

simulating classically with TNS we can reach $\delta \sim 1/\log N$

Lu et al. PRX Q2, 02032 (2021); Yang et al PRB 106, 024307 (2022)

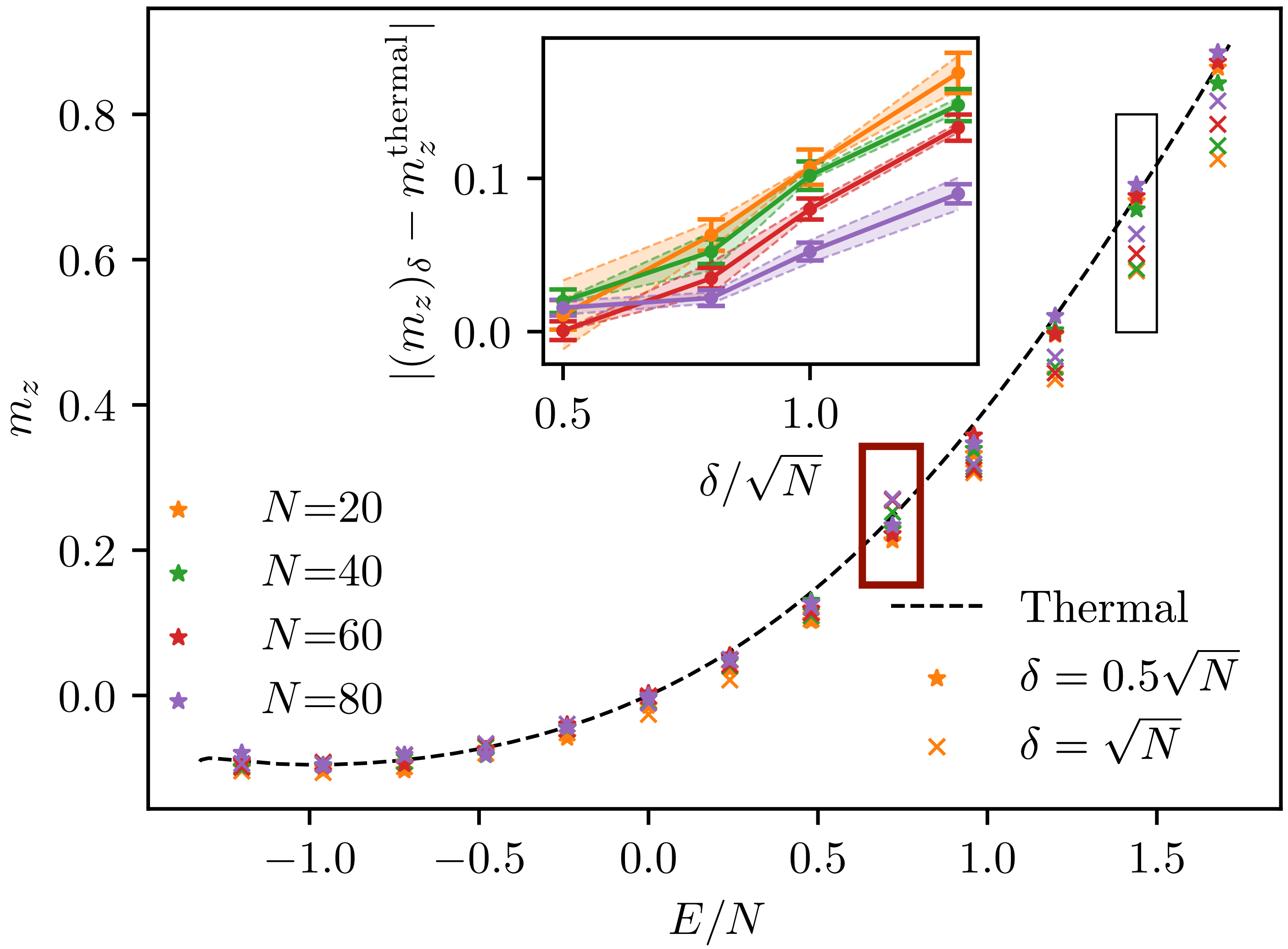
weak ETH probe: diagonal part

$$O_{\alpha\beta} = O(\bar{E})\delta_{\alpha\beta} + e^{-\frac{S(\bar{E})}{2}} f_O(\bar{E},\omega) R_{\alpha\beta}$$

converge to thermal for large systems

TNS simulation

non-integrable quantum Ising chain



microcanonical properties

average magnetization

Yang et al PRB 106, 024307 (2022)

more challenging: off-diagonal part of ETH

$$O_{\alpha\beta} = O(\bar{E})\delta_{\alpha\beta} + e^{-\frac{S(\bar{E})}{2}} f_O(\bar{E},\omega) R_{\alpha\beta}$$

structure function

Luitz, Bar Lev, PRL2016; Mondaini, Rigol 2017; Brenes et al PRL2020, PRB 2020…;
Schönle et al PRB2021; Essler, de Klerk, 2307.12410; ….

filter ensemble as ETH probe

function $f_O(\bar{E}, \omega)$ related to two-time correlators

Luo et al 2312.00736

filter ensemble as ETH probe

function $f_O(\bar{E}, \omega)$ related to two-time correlators

$$C_O(t) = \mathrm{tr}\left(\rho_E O(t) O(0)\right)$$

$$S_O(\omega) = \sum_{\alpha\beta} \rho_{\alpha\alpha} |O_{\alpha\beta}|^2 \delta(\omega - E_\beta + E_\alpha)$$

Luo et al 2312.00736

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Luo et al 2312.00736

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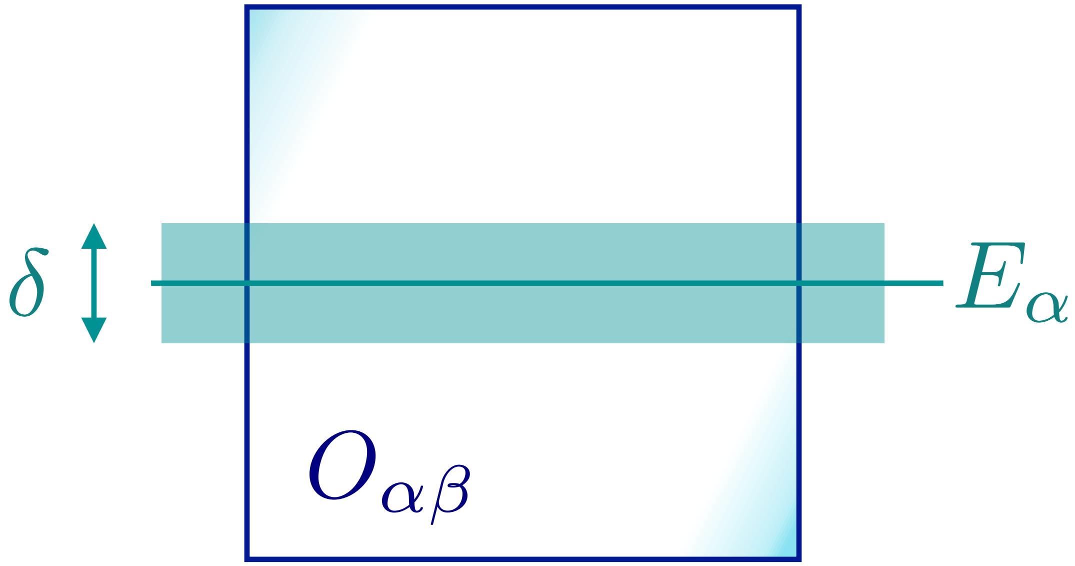

Luo et al 2312.00736

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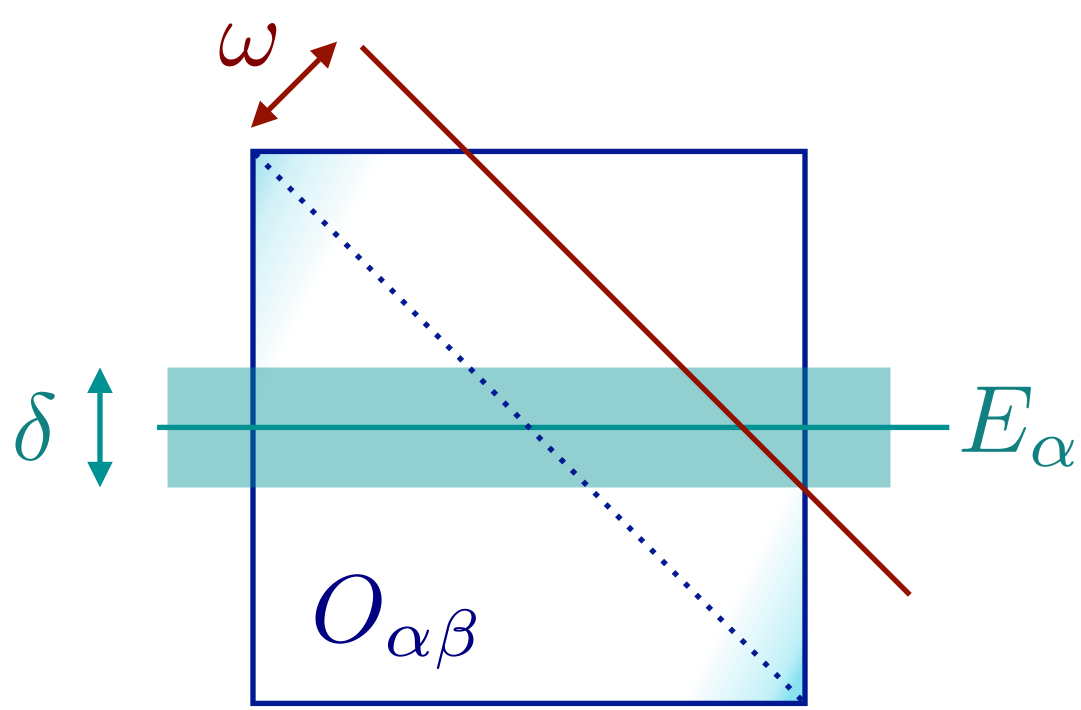

Luo et al 2312.00736

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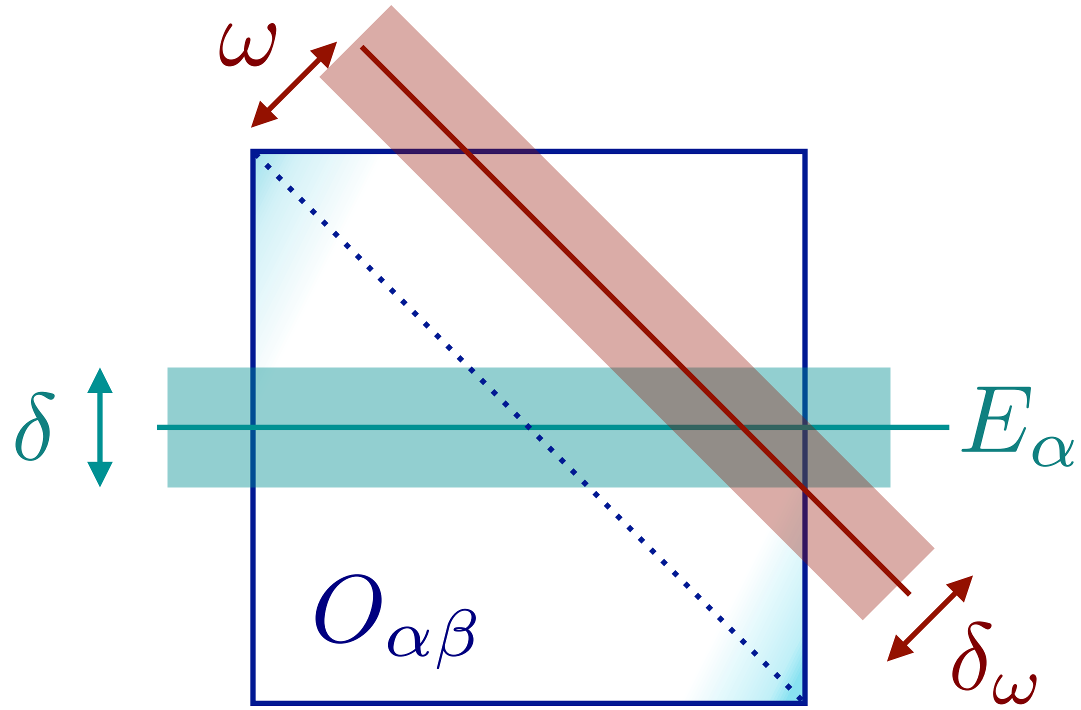


$P_{\delta_\omega}(\omega)$

filter in energy difference (commutator)

Luo et al 2312.00736

filter ensemble as ETH probe

function $f_O(\bar{E}, \omega)$ related to two-time correlators

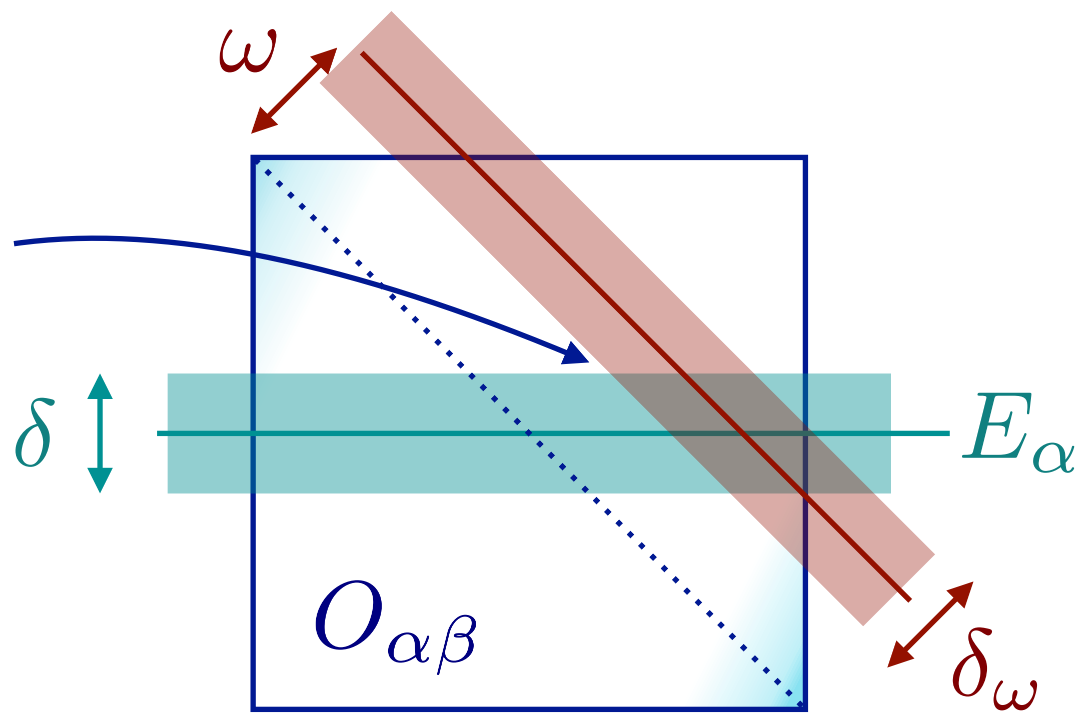
$$C_O(t) = \mathrm{tr}\left(\rho_E O(t) O(0)\right)$$

for filter ensemble

$$S_O(\omega) = \sum_{\alpha\beta} \rho_{\alpha\alpha} |O_{\alpha\beta}|^2 \delta(\omega - E_\beta + E_\alpha)$$

$e^{S(E+\omega)} |O_{E,E+\omega}|^2$

average times density of states factor

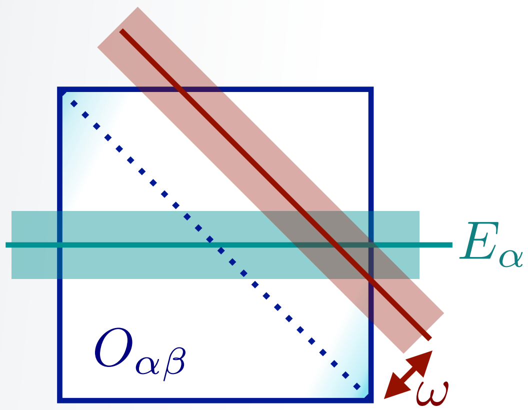


$P_{\delta_\omega}(\omega)$

filter in energy difference (commutator)

Luo et al 2312.00736

filter ensemble as ETH probe



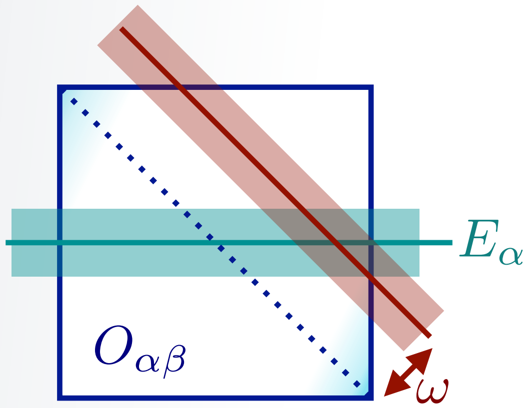
broadened spectral function of filter ensemble

$$S_O^\rho(\omega) \approx e^{S(E+\omega)} |O_{E,E+\omega}|^2$$

see also Pappalardi, Foini, Kurchan, 2304.10948

Luo et al 2312.00736

filter ensemble as ETH probe



broadened spectral function of filter ensemble

$$S^{\rho}_{O}(\omega) \approx e^{S(E+\omega)} |O_{E,E+\omega}|^2$$

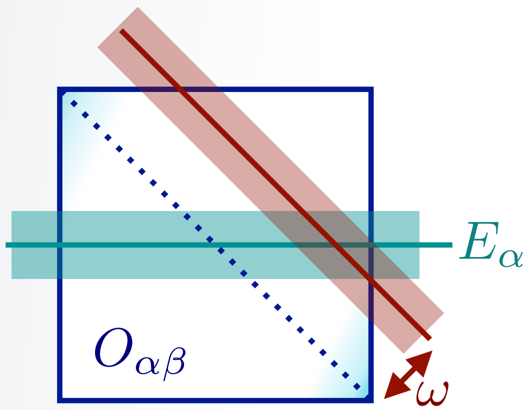
$$\approx e^{\frac{S(E+\omega)-S(E)}{2}} |f_O(E+\omega/2,\omega)|^2$$

using ETH

see also Pappalardi, Foini, Kurchan, 2304.10948

Luo et al 2312.00736

filter ensemble as ETH probe



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$$\approx e^{\frac{S(E+\omega)-S(E)}{2}} |f_O(E+\omega/2,\omega)|^2$$

using ETH

entropy factor extracted from DoS calculation or eliminated from $S_O^\rho(-\omega)$

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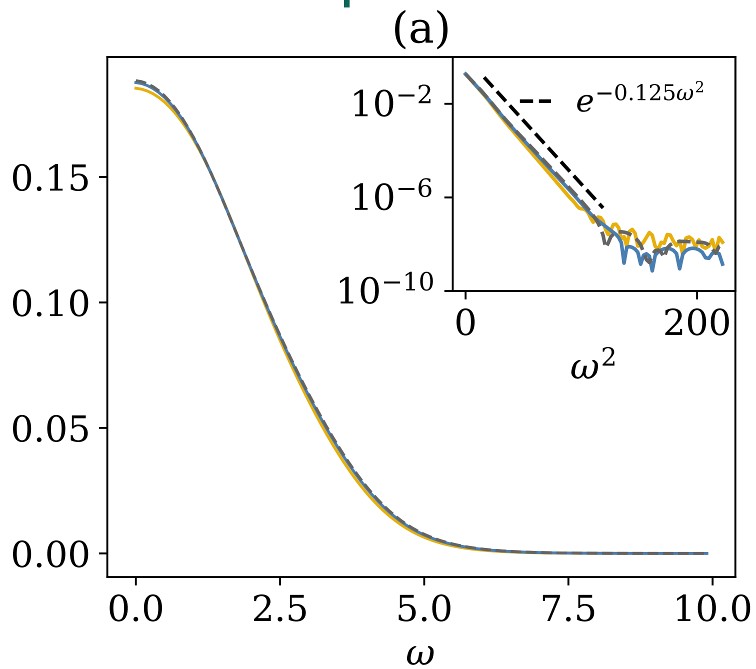
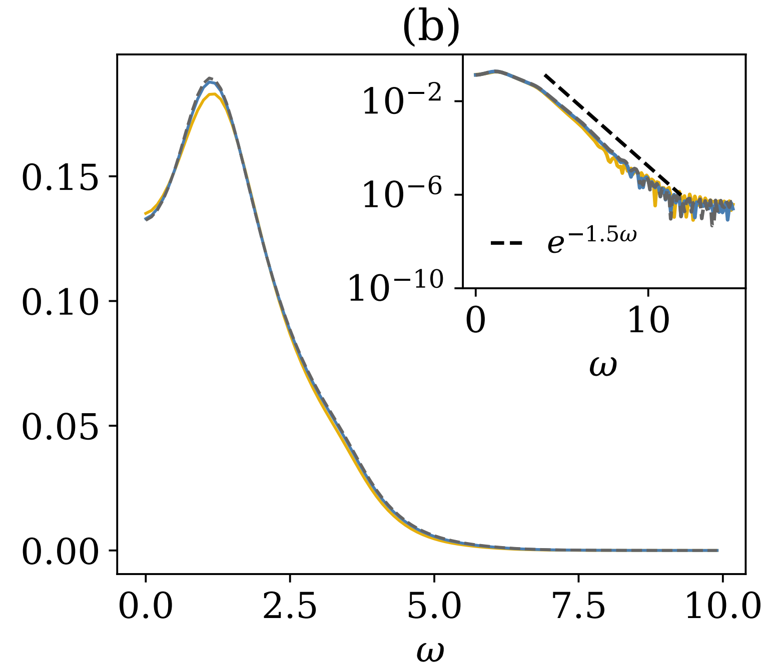
ω-dependence of $|f_O(E,\omega)|^2$

$N = 20 - 60$
$E/N = 0.5$

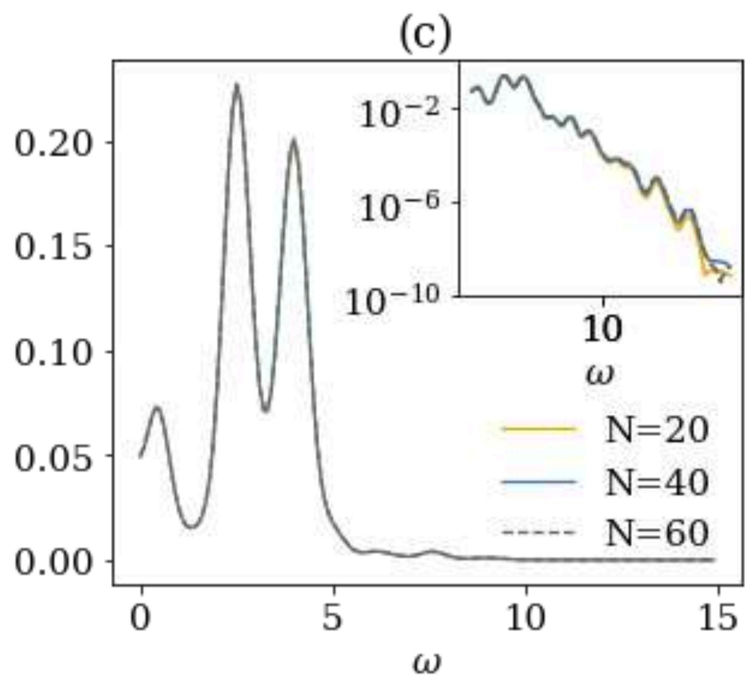
integrable



non-integrable



disordered



converged in size
asymptotic behaviour

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To conclude

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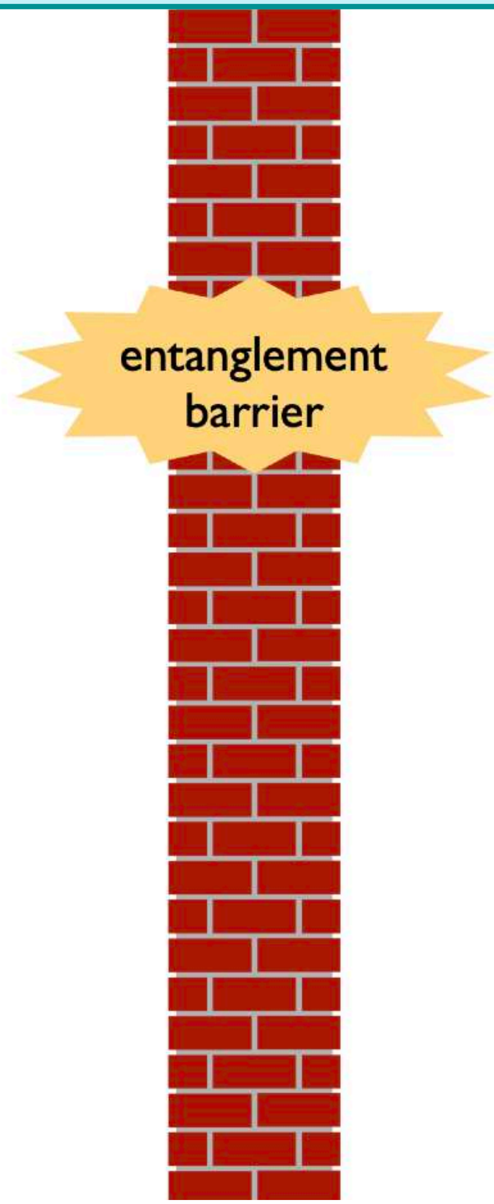
energy filters & TNS can provide other (classical and quantum) tools to get dynamical properties



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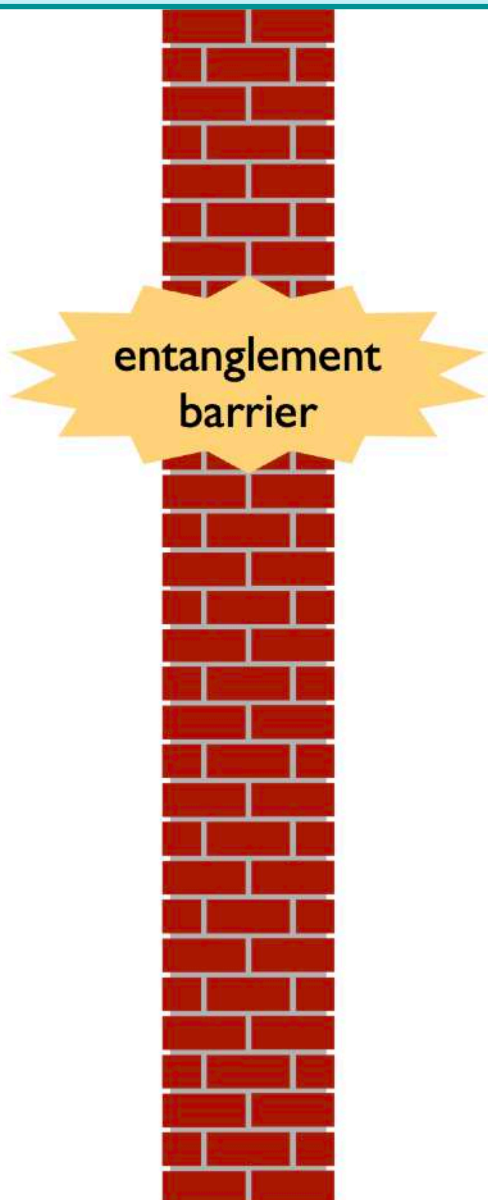

spectral properties of a QMB Hamiltonian

Yang, Iblisdir, Cirac, MCB, PRL124, 100602 (2020)
Lu, PRX Quantum 2, 020321 (2021)
Yang, Cirac, MCB, PRB 106, 024307 (2022)
Luo, Trivedi, MCB, Cirac, PRB 109, 134304 (2024)

further possibilities: apply to non-ergodic systems, probe fluctuation-dissipation relations, further explore off-diagonal matrix elements…

MCQST

Thanks for your attention!

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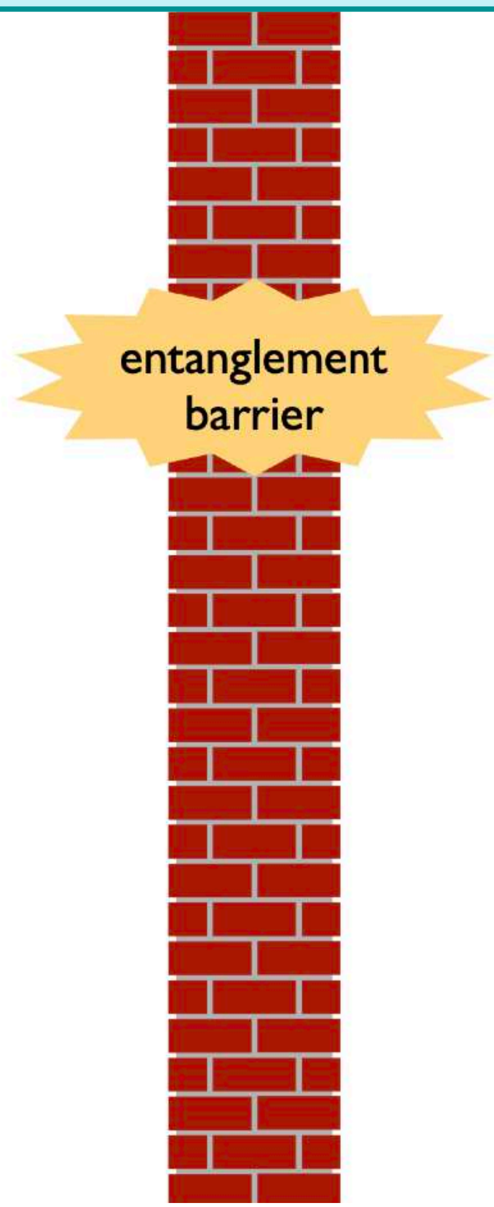

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