Universal distribution of the number of minima for random walks and Lévy flights

Grégory Schehr
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Journées de Physique Statistique
Paris, January 30-31, 2025

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- Anupam Kundu (ICTS-TIFR, Bengaluru)
- Satya N. Majumdar (LPTMS, Univ. Paris-Saclay)

A. Kundu, S. N. Majumdar, G. S., Phys. Rev. E **110**, 024137 (2024)

A. Kundu, S. N. Majumdar, G. S., J. Phys. A: Math. Theor. **58**, 035002 (2025)

Outline

- Motivations and background
- Main results
- Sketch of the derivation
- Conclusion and perspectives

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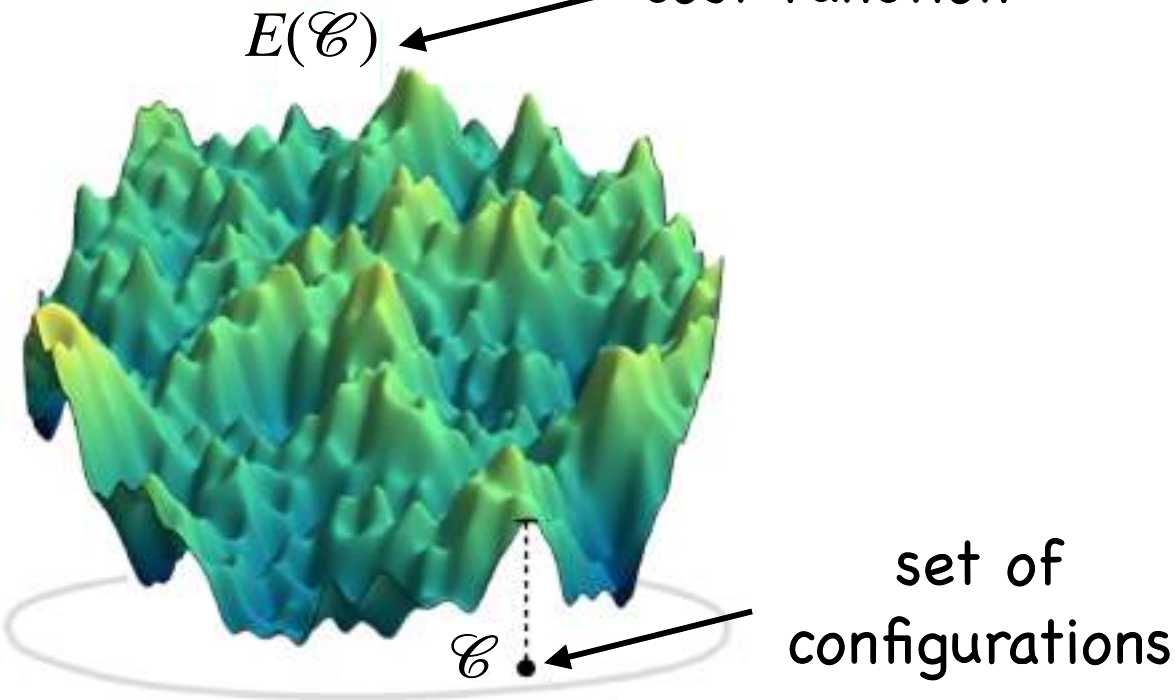
Motivations

- Study of random landscapes



from V. Ros

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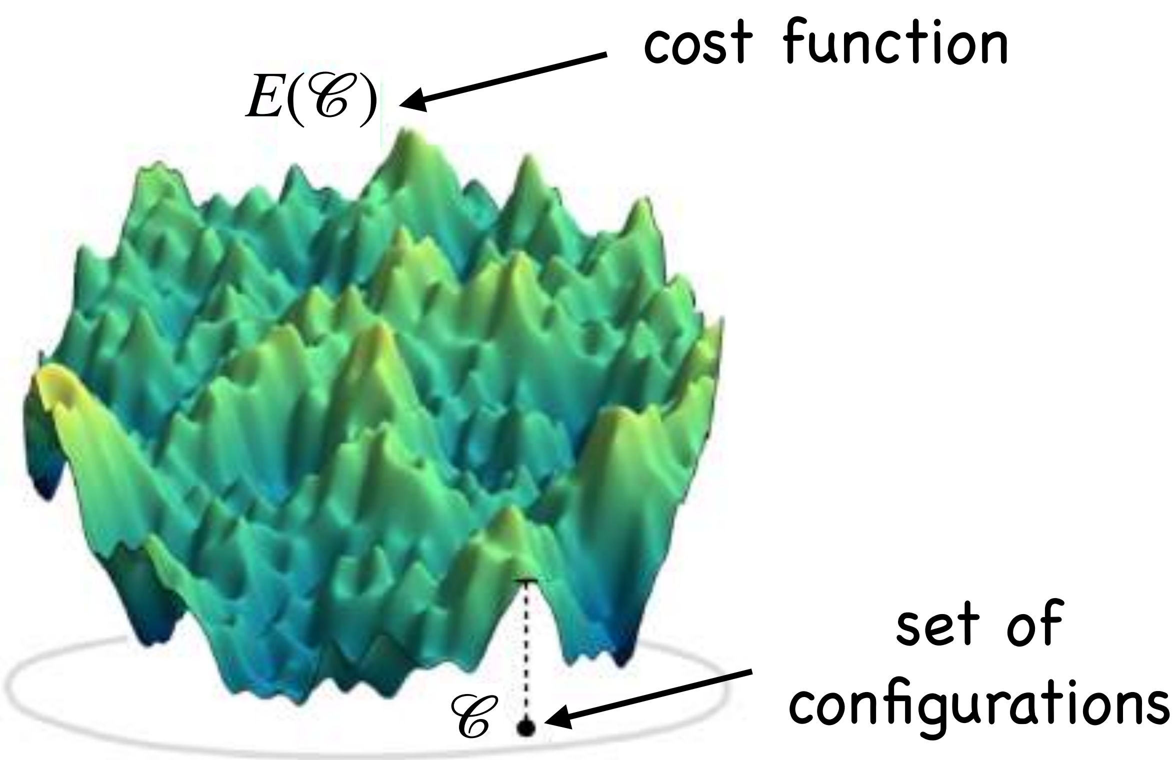

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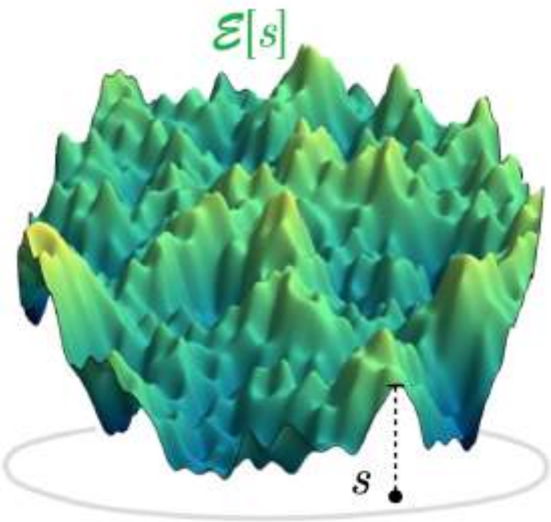
Q: what is the number of stationary points ?

Motivations



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- Natural and important question in various contexts

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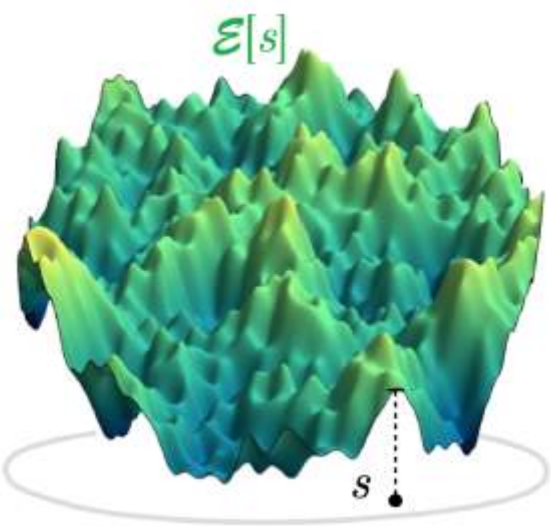

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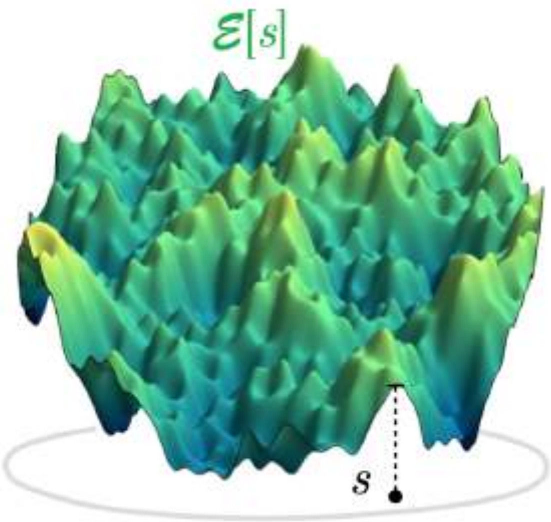

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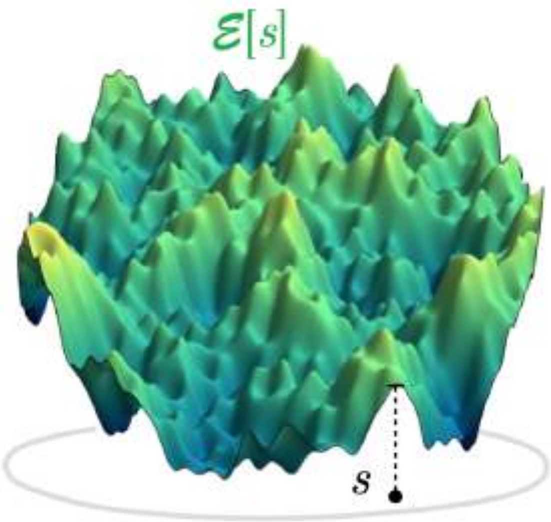

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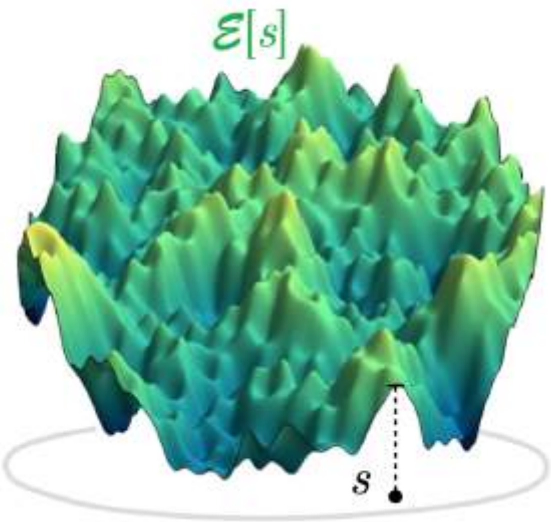

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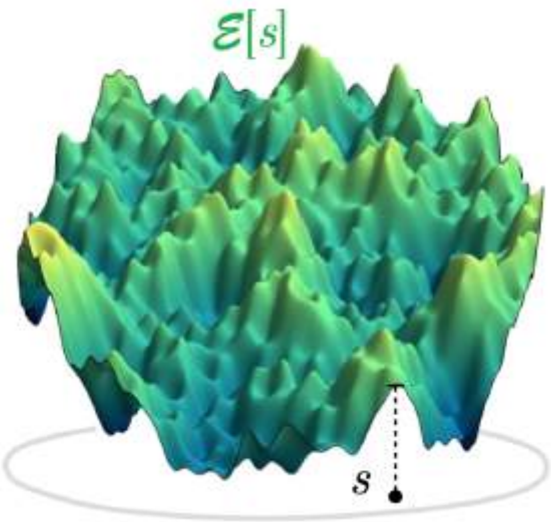

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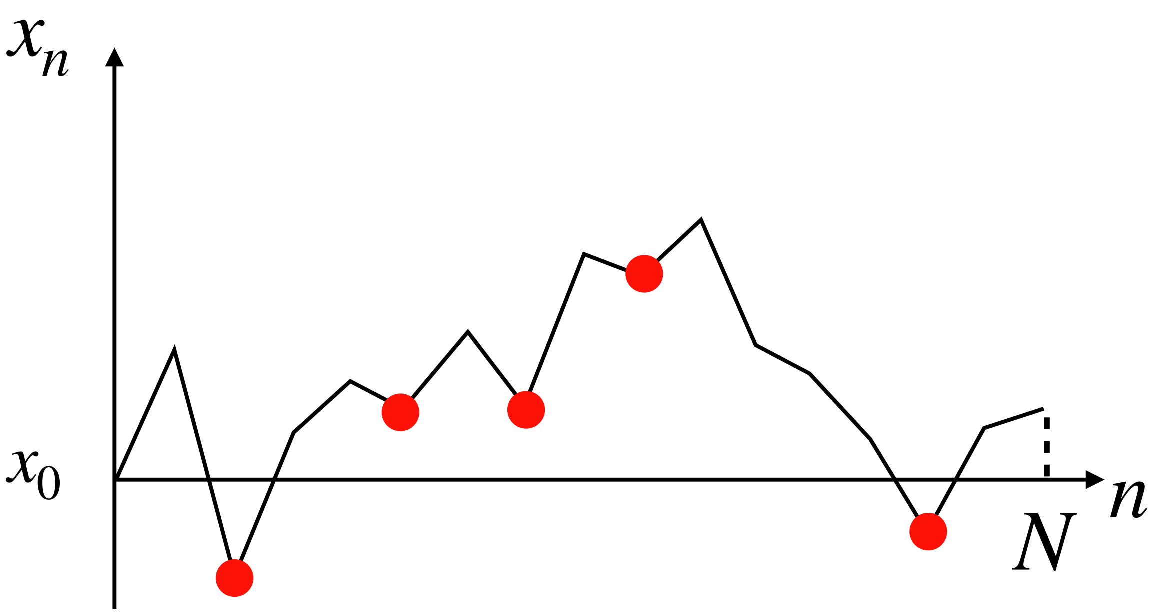
  - ... also relevant for one-dimensional landscapes !

Motivations

- Here: one-dimensional random landscape generated by a random walk

$$x_0 = 0 \quad , \quad x_n = x_{n-1} + \eta_n \, , \, n \geq 1$$

IID rand. var. distributed with $\phi(\eta)$, continuous and symmetric



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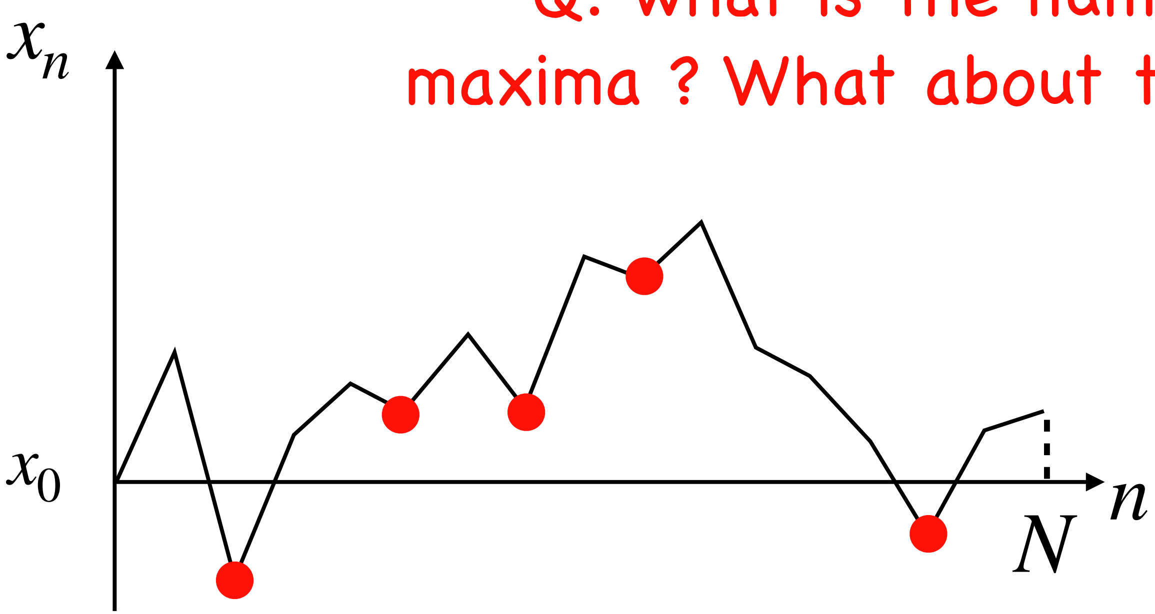
Q: what is the number of minima, maxima ? What about their correlations ?



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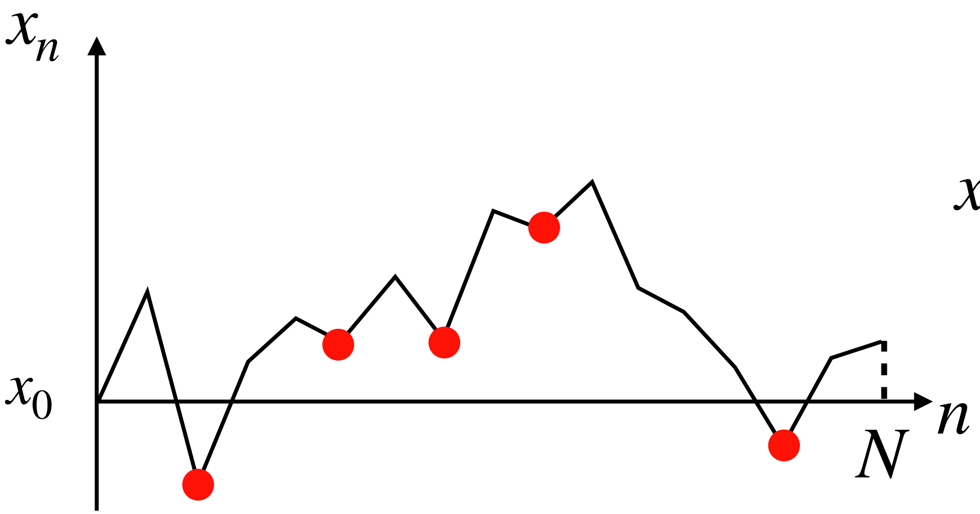

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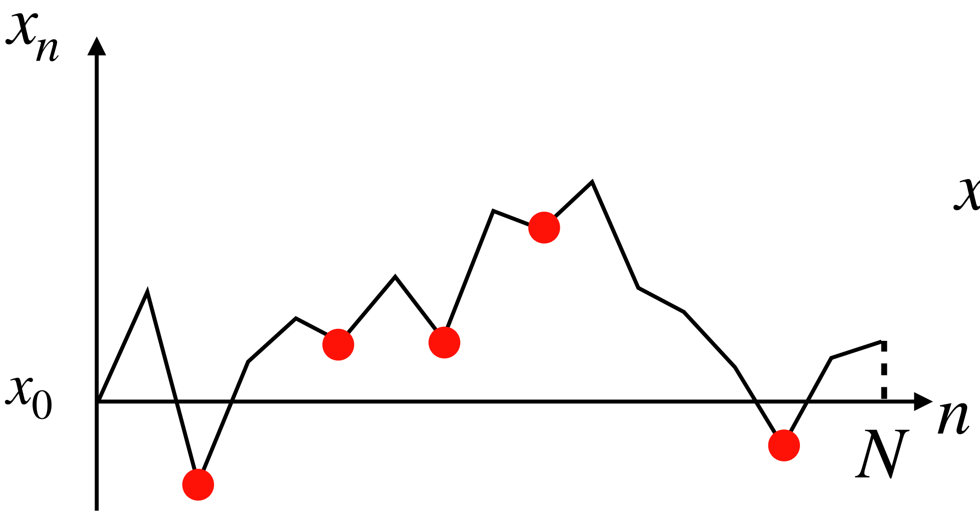

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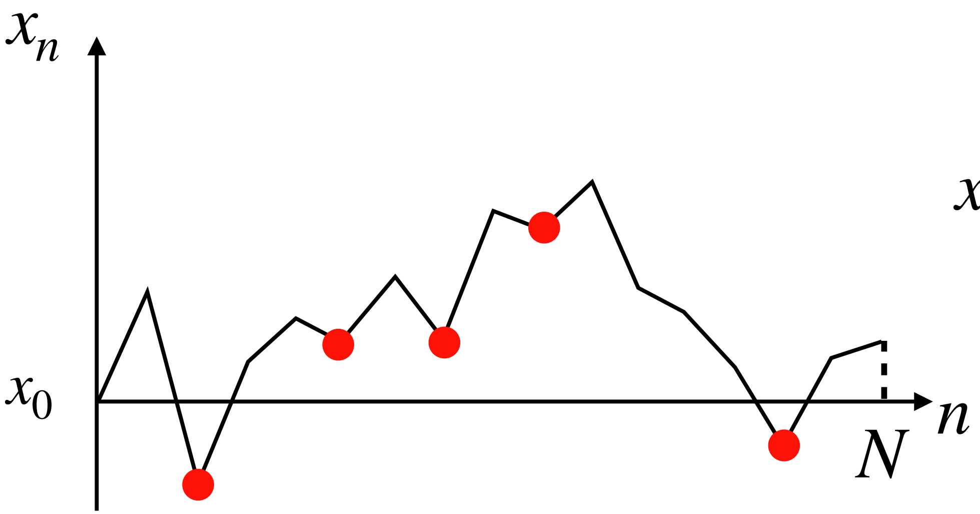

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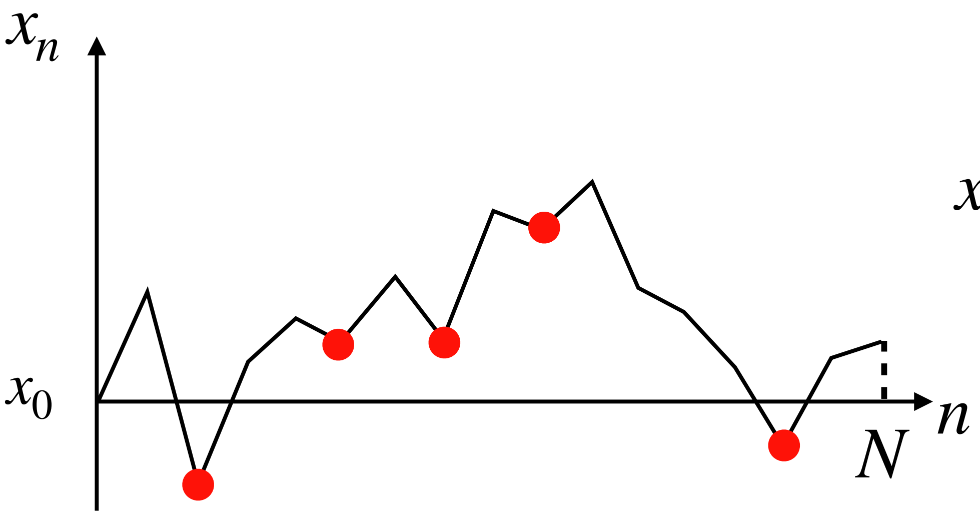

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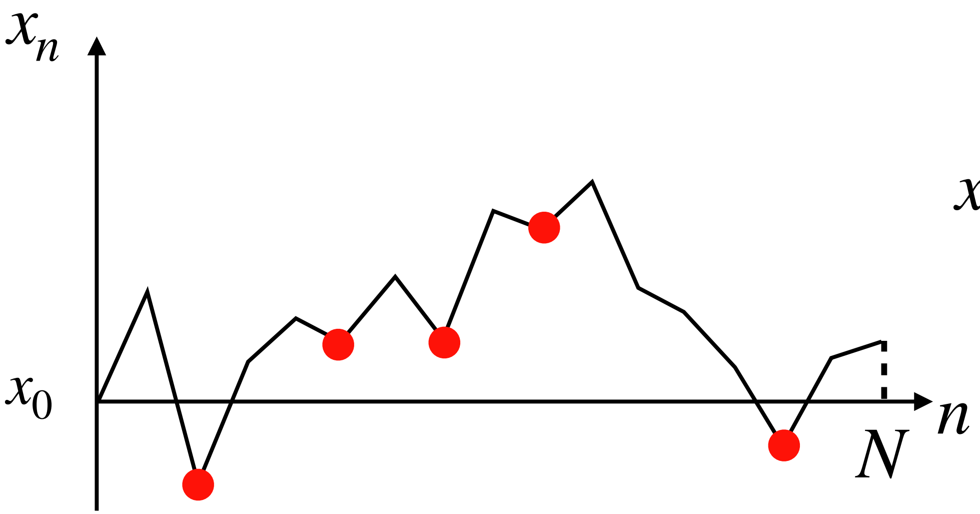

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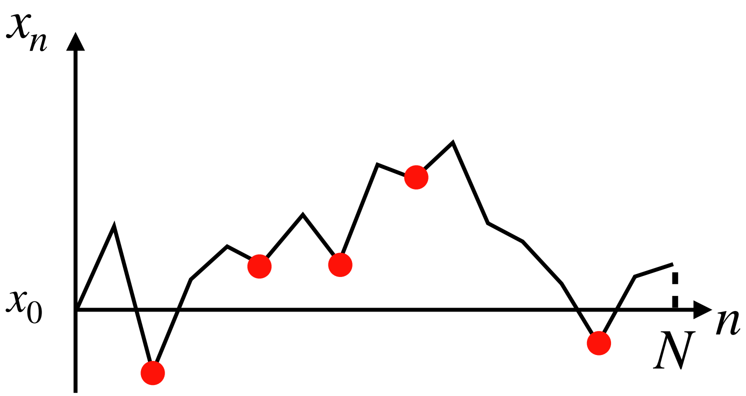

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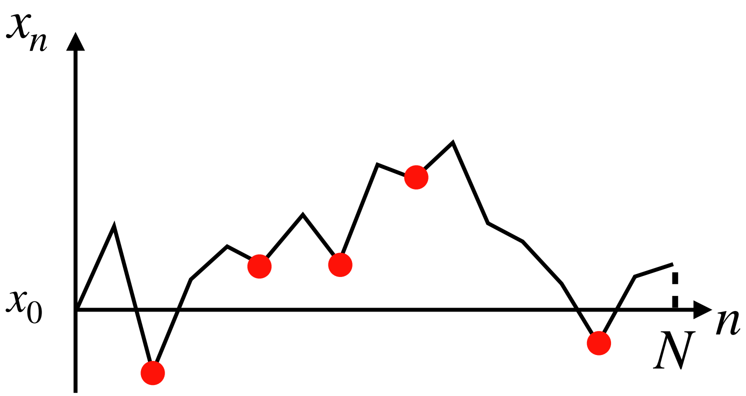

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PHYSICAL REVIEW E VOLUME 62, NUMBER 1 JULY 2000

Extremal-point densities of interface fluctuations

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[1]Department of Physics, University of Maryland, College Park, Maryland 20742-4111
[2]Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435
[3]Supercomputer Computations Research Institute, Florida State University, Tallahassee, Florida 32306-4130
[4]Center for Nonlinear Science and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87544

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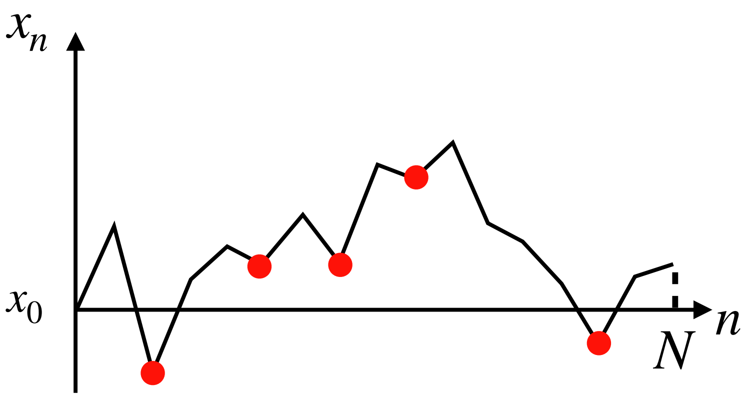

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⇨ results only for the average number of minima, but nothing on the fluctuations !

Motivations

- These questions have been widely studied for lattice paths (or equivalently Bernoulli random walks) using combinatorial approaches

Narayana, Krattenthaler, Stanley, ...

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- A different terminology is often used in this context:
  - « turns » stand for « stationary points »
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- More recent works considered constrained lattice paths

Asymptotics of Bernoulli random walks, bridges, excursions and meanders with a given number of peaks

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- Exact results for finite number of steps N + central limit theorem in the limit $N \to \infty$

$$\mathbb{E}(\mathrm{m}_N) = N/4 + O(1) \quad , \quad \mathrm{Var}(\mathrm{m}_N) = N/16 + O(1)$$

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This talk: what about more general constrained random walks with continuous jump/increment distribution ?

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- Motivations and background
- Main results
- Sketch of the derivation
- Conclusion and perspectives

Main results

A. Kundu, S. N. Majumdar, G. S.

- Random walks (RWs) on the line $x_0 = 0 \quad , \quad x_n = x_{n-1} + \eta_n \, , n \geq 1$

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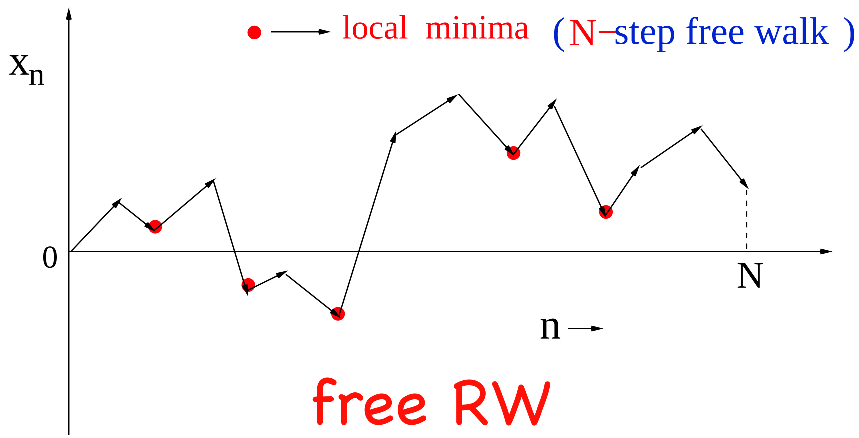

free RW

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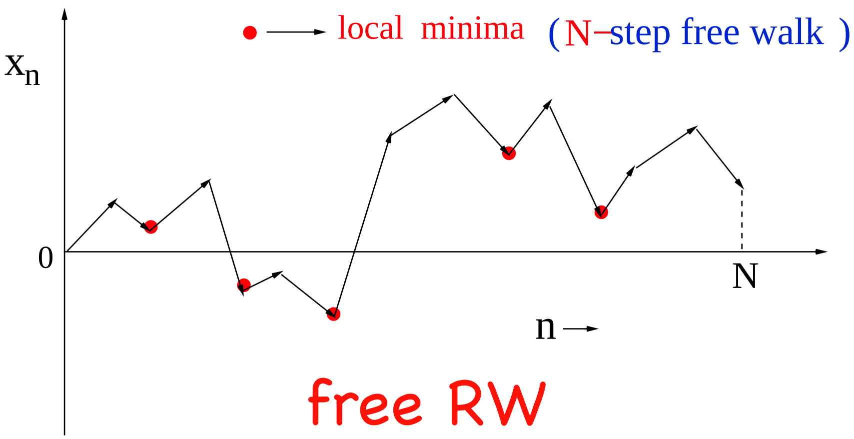

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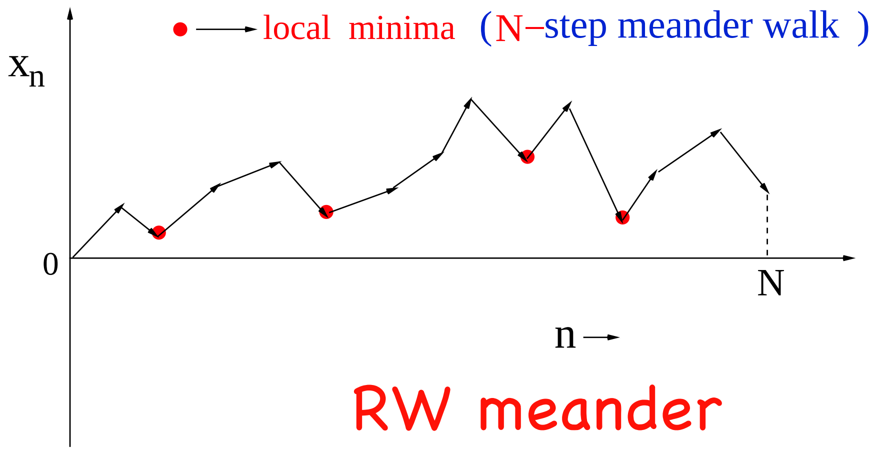

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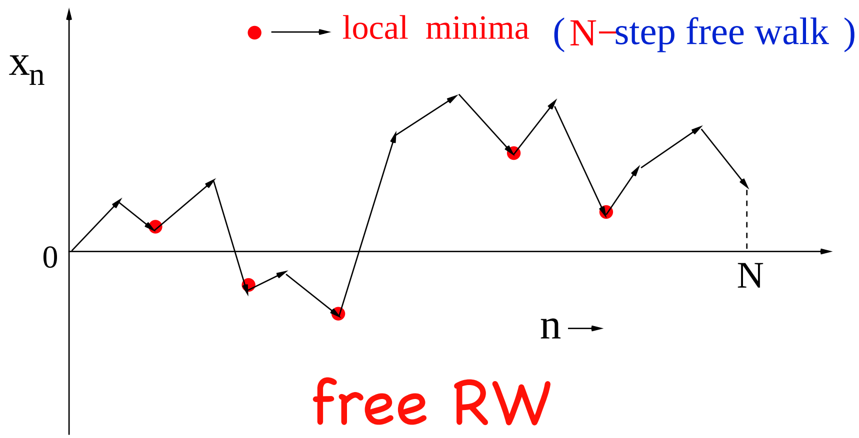

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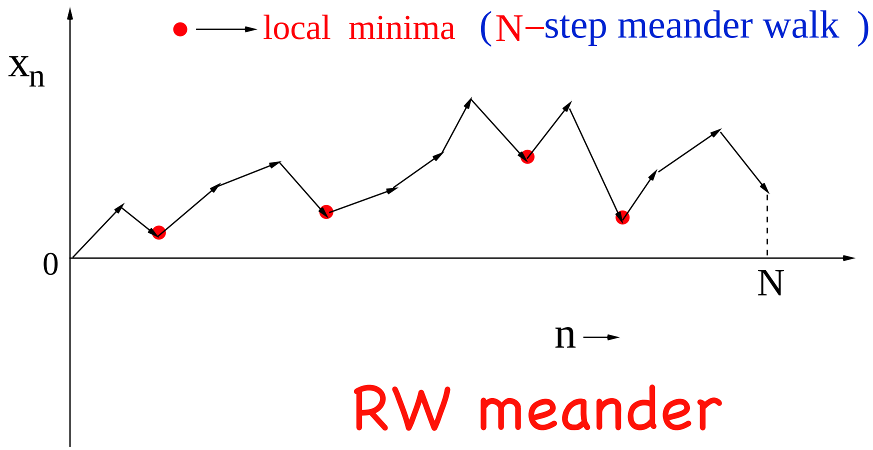
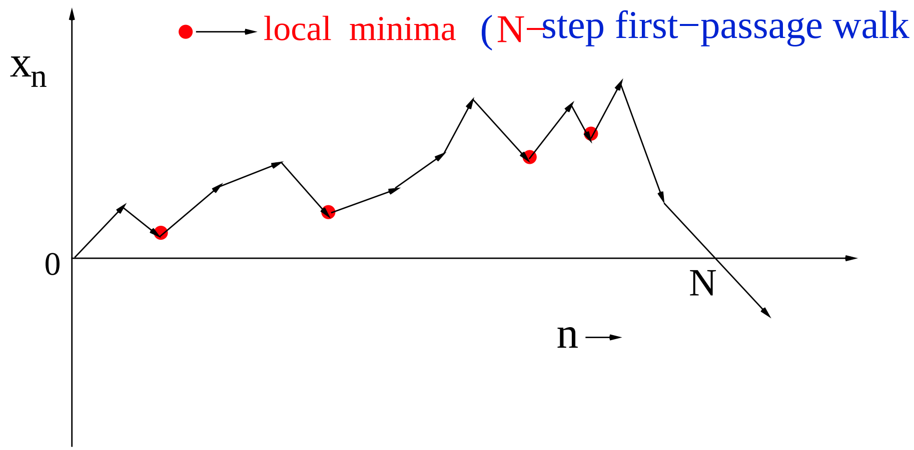


RW meander

- As a corollary

RW up to the 1st-passage at the origin



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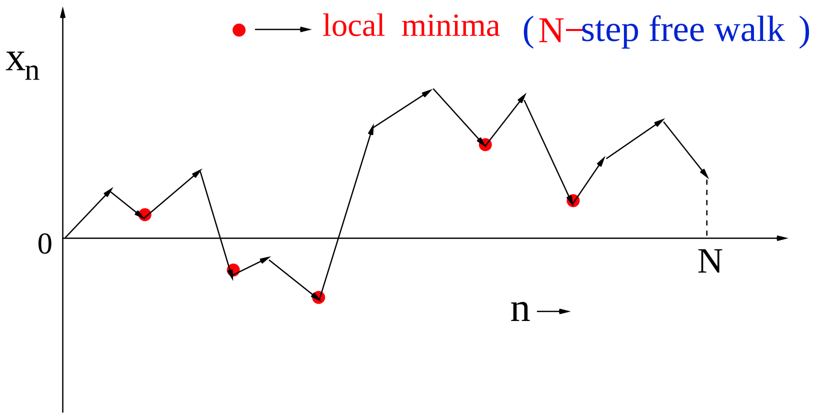

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- Stat. of the number of minima m_N^{rw} for free random walks of N steps

$$Q^{rw}(m,N) = \text{Prob}\,.(m_N^{rw} = m) = \begin{cases} \dfrac{1}{2^N}\dbinom{N+1}{2m+1}, & \text{for} \quad 0 \leq m \leq \dfrac{N}{2} \\ 0\,, & \text{for} \quad m > \dfrac{N}{2} \end{cases}$$

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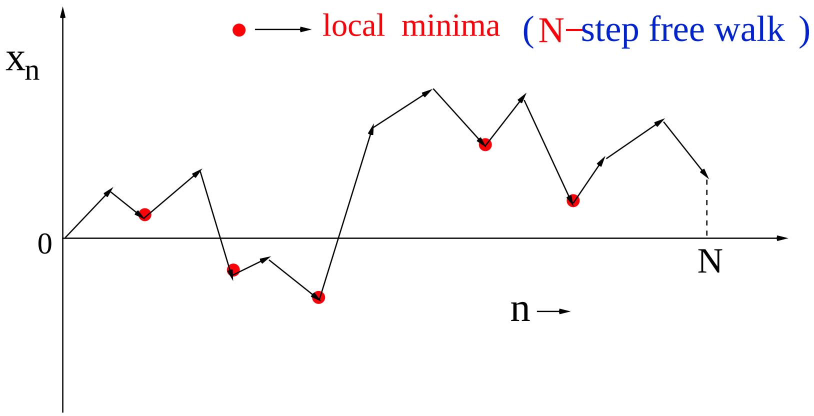

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(Obviously) universal, i.e., independent of the increment/ jump distribution $\phi(\eta)$

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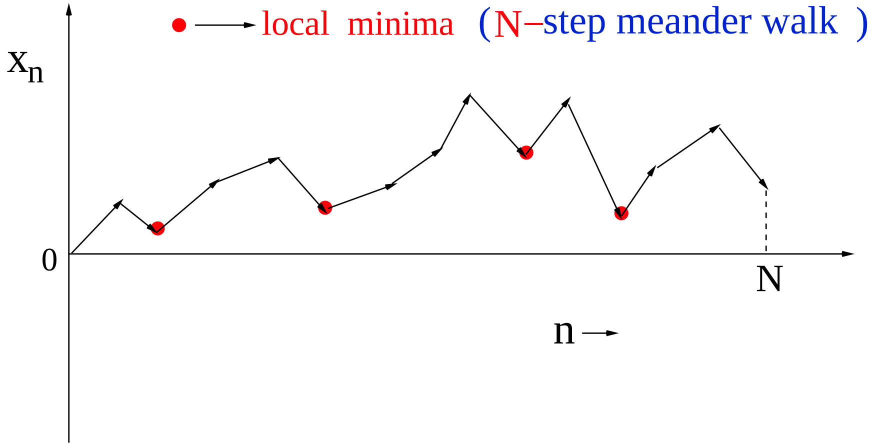

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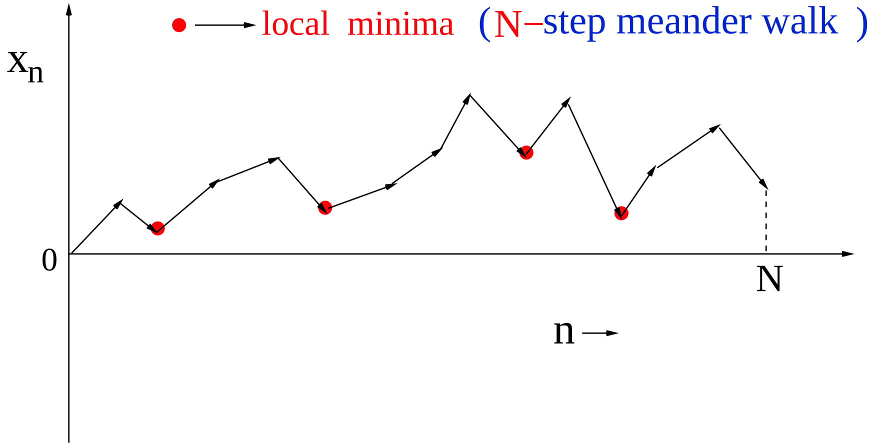

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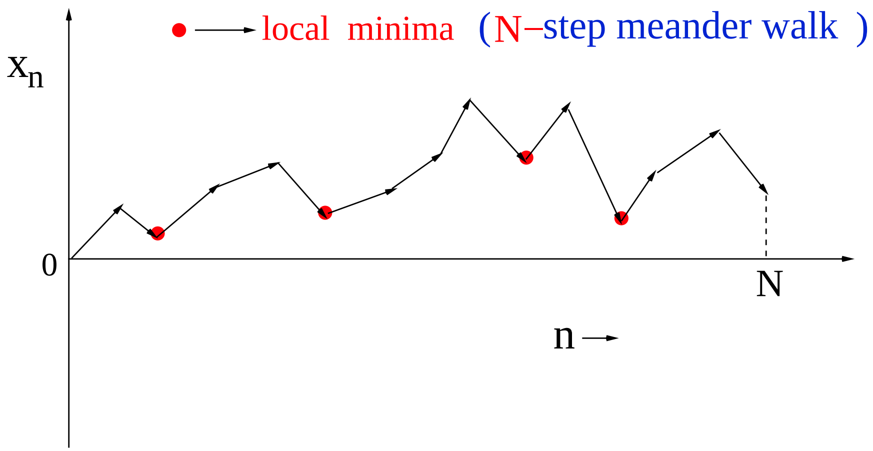

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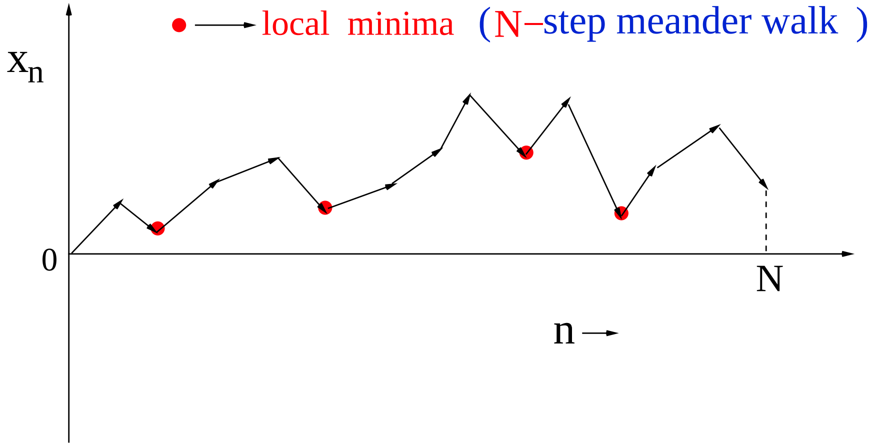

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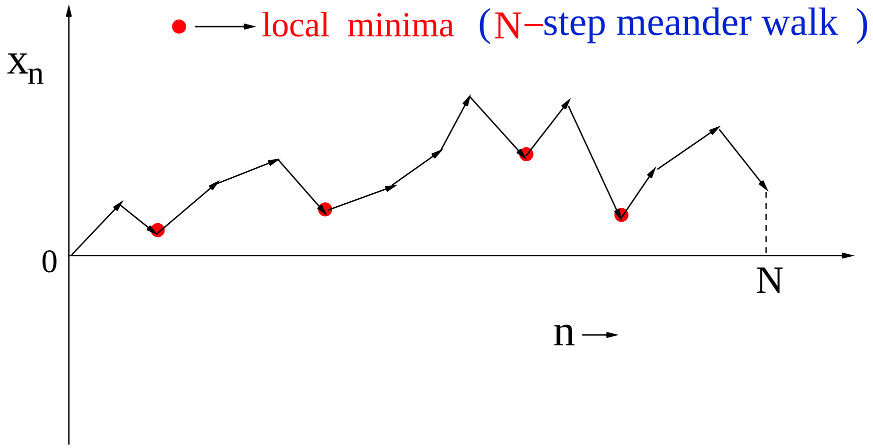
$$Q^{\rm me}(m,N) = \begin{cases} \dfrac{(N+1)\,(N-1)!}{2^{2m+N+1}\,(m\,!)^2\,(m+1)\,(N-2m-1)!} & \text{for} \quad 0 \leq m \leq \frac{N}{2} - 1\,, \\ 0 & \text{for} \quad m > \frac{N}{2} - 1 \end{cases}$$

(Less obviously) universal, i.e., independent of the increment/jump distribution $\phi(\eta)$!

... Generalization of Sparre Andersen theorem

A reminder on Sparre Andersen theorem



$$x_0 = 0 \quad , \quad x_n = x_{n-1} + \eta_n \, , \, n \geq 1$$

IID rand. var. distributed with $\phi(\eta)$, continuous and symmetric

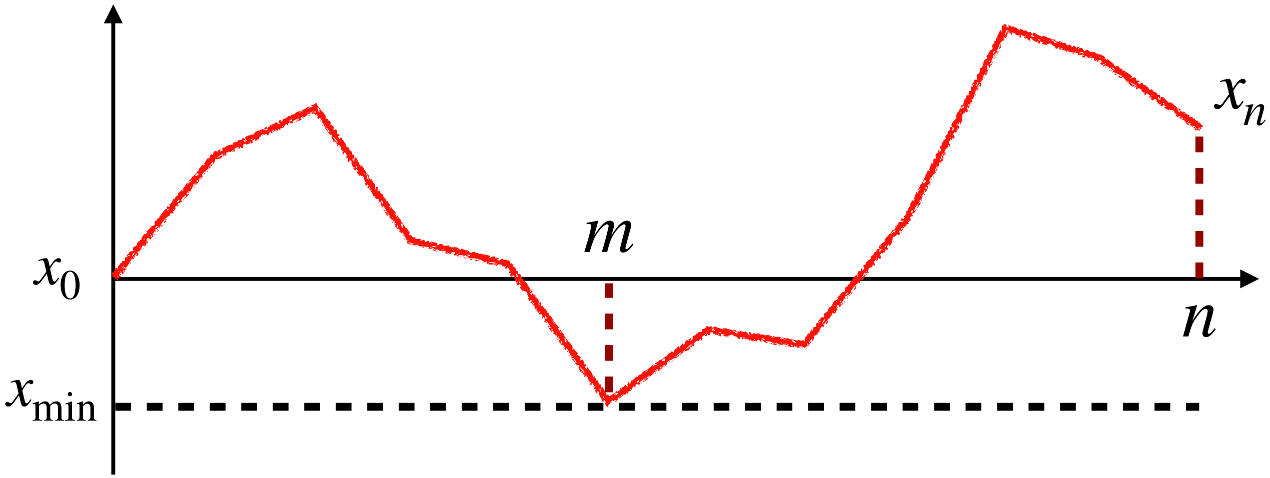
- Sparre Andersen theorem (1954) for the « survival » probability

$$q_N = \text{Prob}\,.\,(x_1 \geq 0, x_2 \geq 0, \cdots, x_N \geq 0) = \frac{1}{2^{2N}} \binom{2N}{N}$$

Universal, i.e., independent of the increment/jump distribution $\phi(\eta)$

A simple proof of the Sparre Andersen theorem

Ph. Mounaix, S. N. Majumdar, G. S., J. Phys. A (2020)



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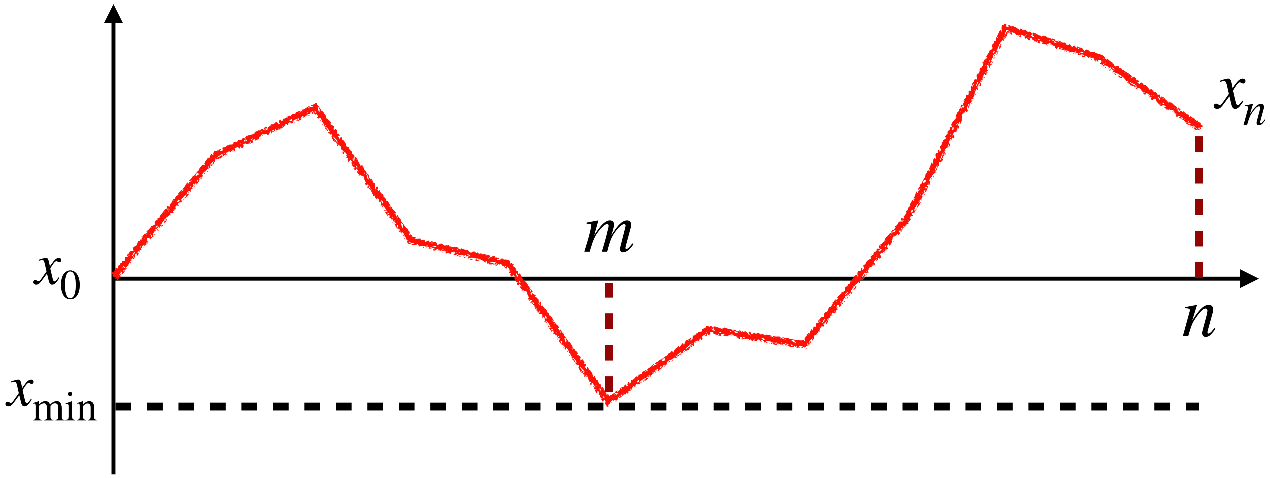

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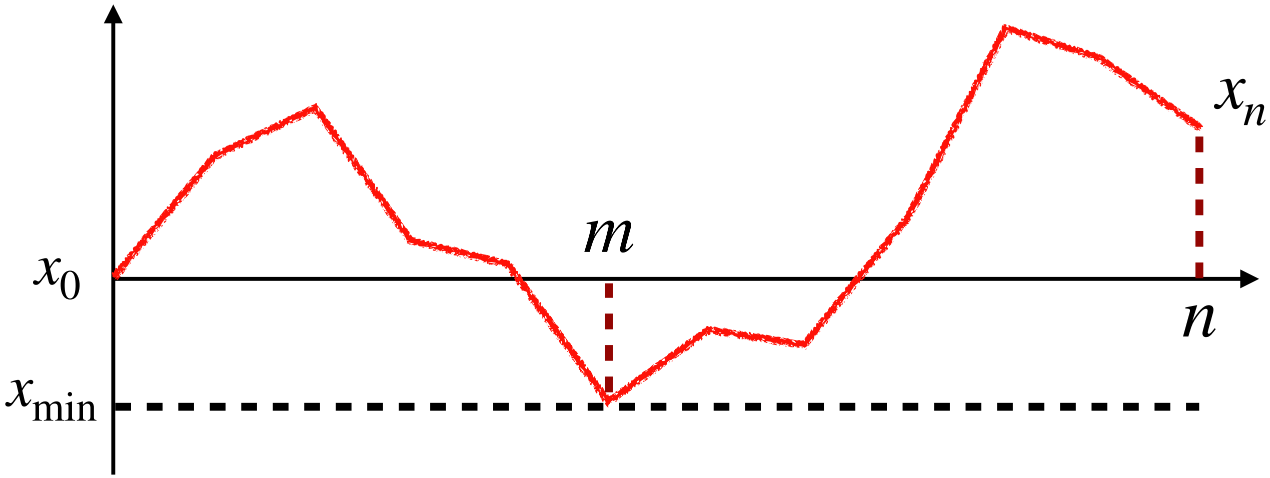

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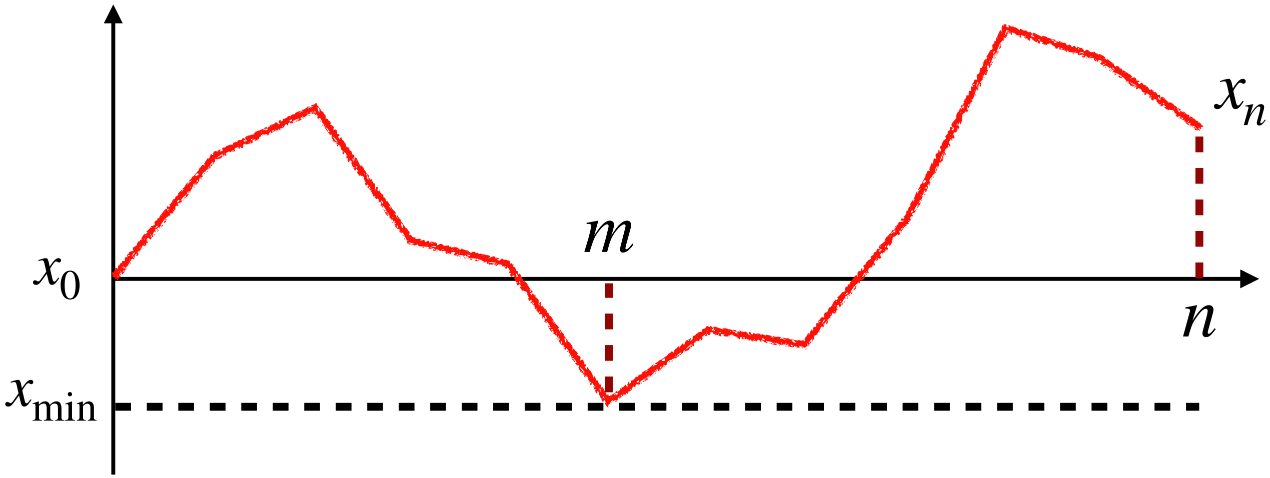

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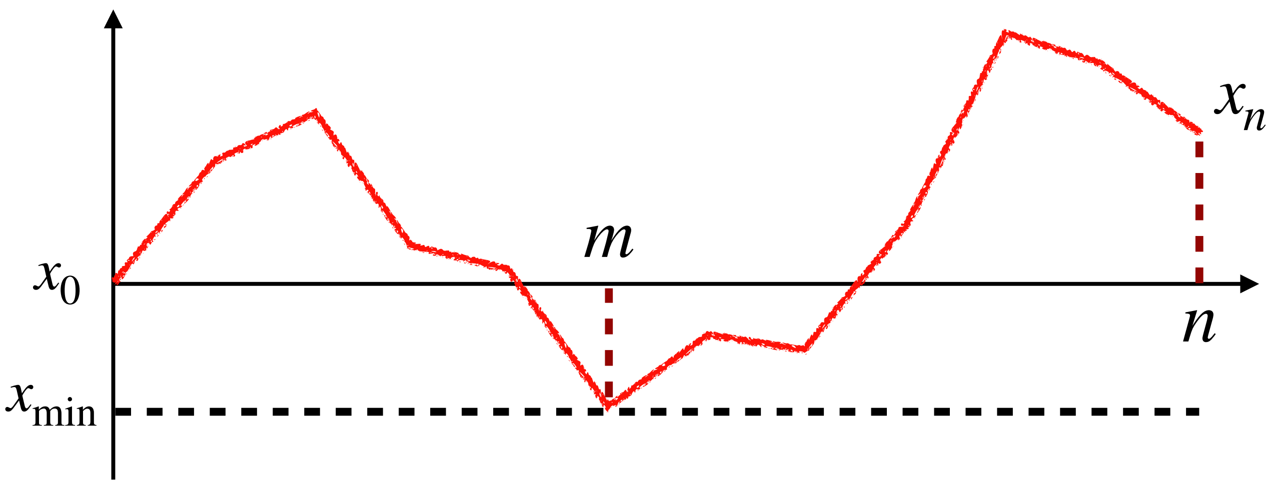

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$$P_n(m) = \text{Prob.}\,(t_{\min} = m) = q(m)q(n-m) \quad , \quad 0 \le m \le n$$

survival proba. up to step $n - m$

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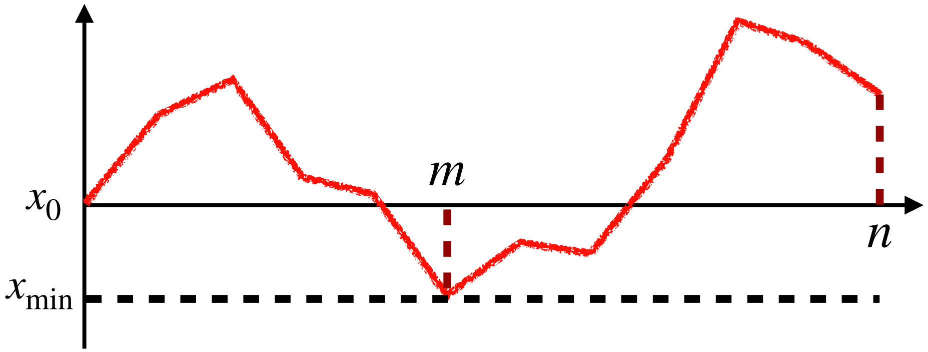

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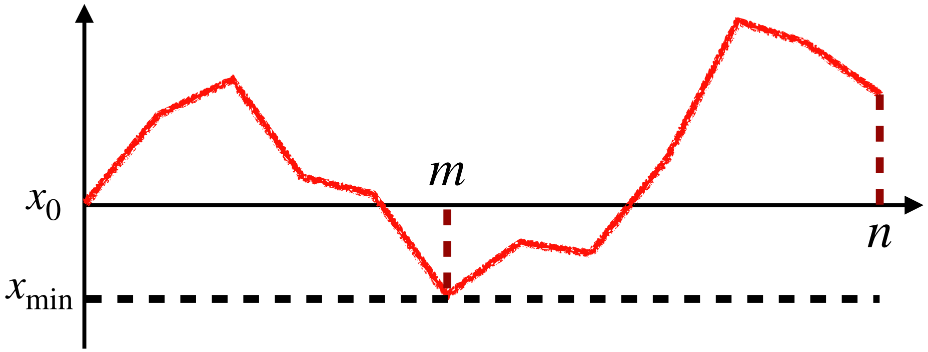

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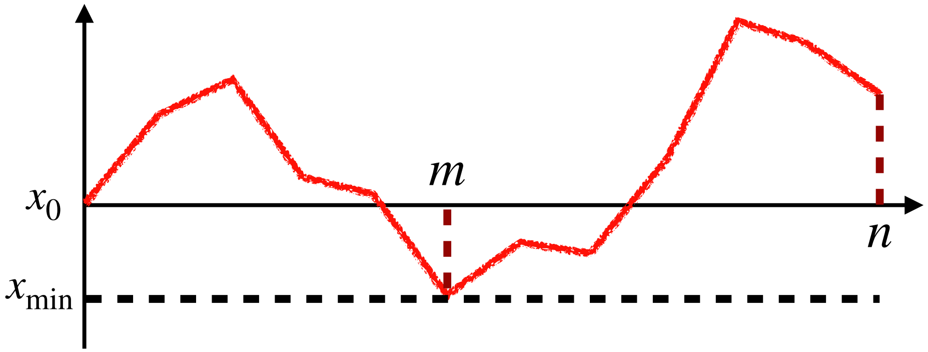

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$$\tilde{q}(z) = \sum_{m \ge 0} z^n q(n)$$

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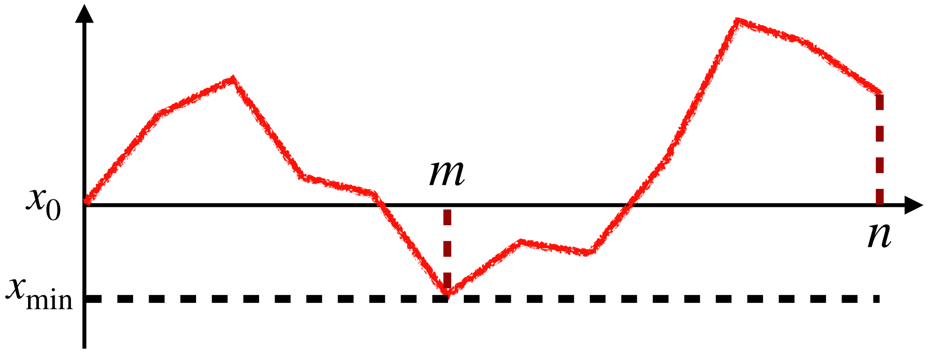

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Back to our main results

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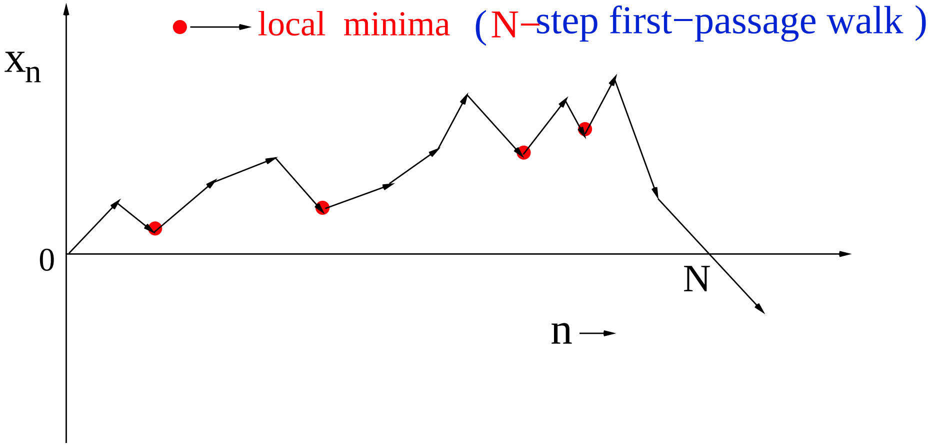

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IID rand. var. distributed with $\phi(\eta)$, continuous and symmetric

- Joint proba. of the number of minima m_N^{fp} and the first-passage time

Back to our main results

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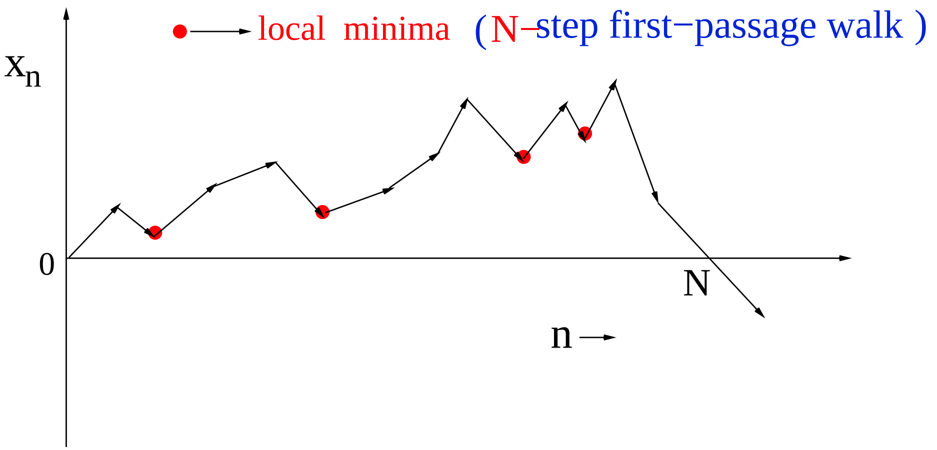

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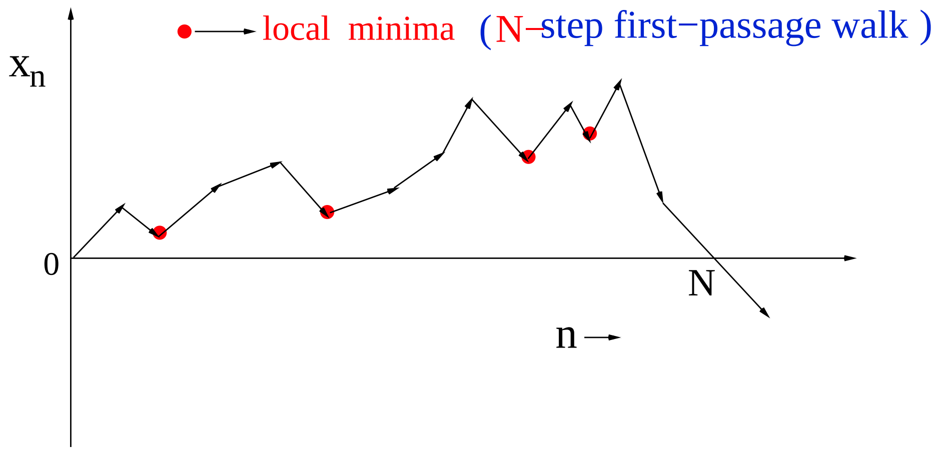

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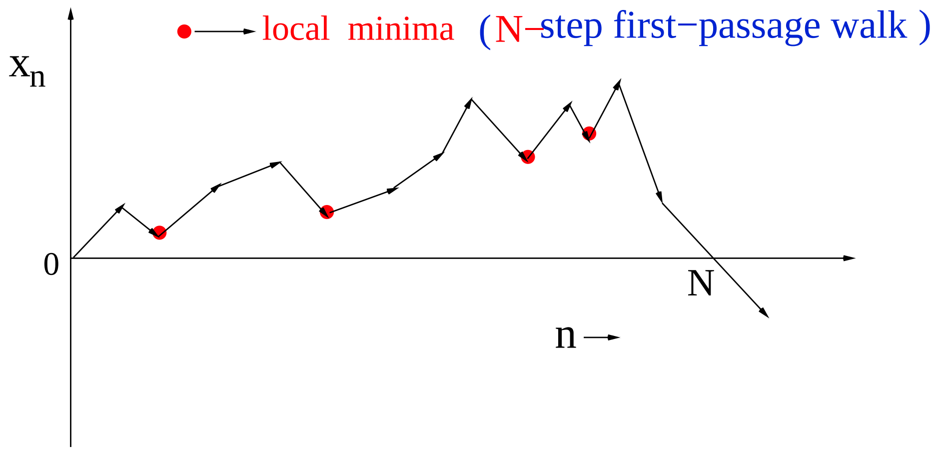

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Main results

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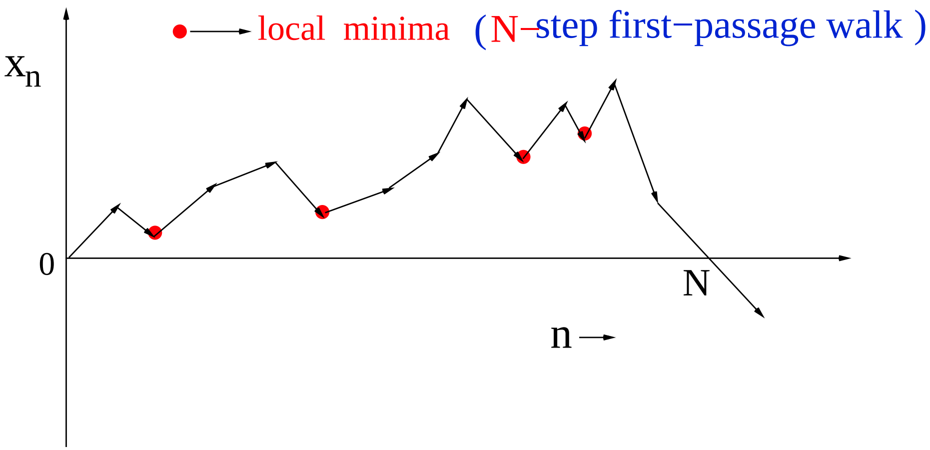

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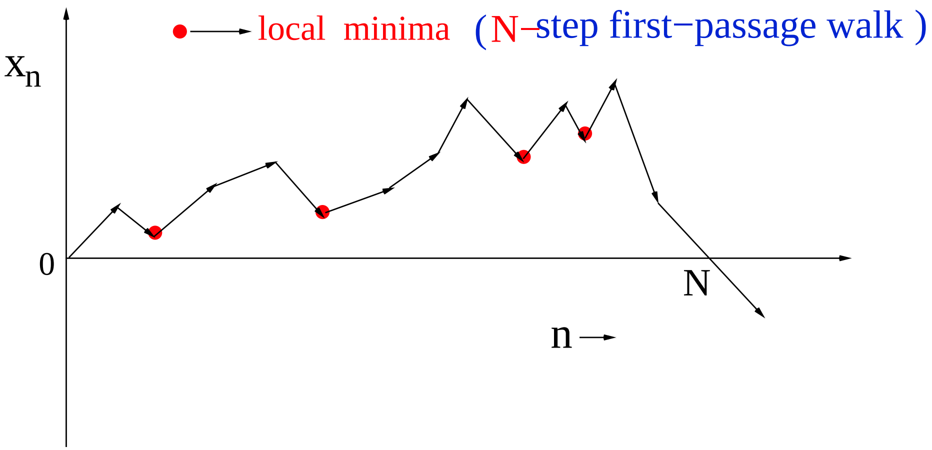

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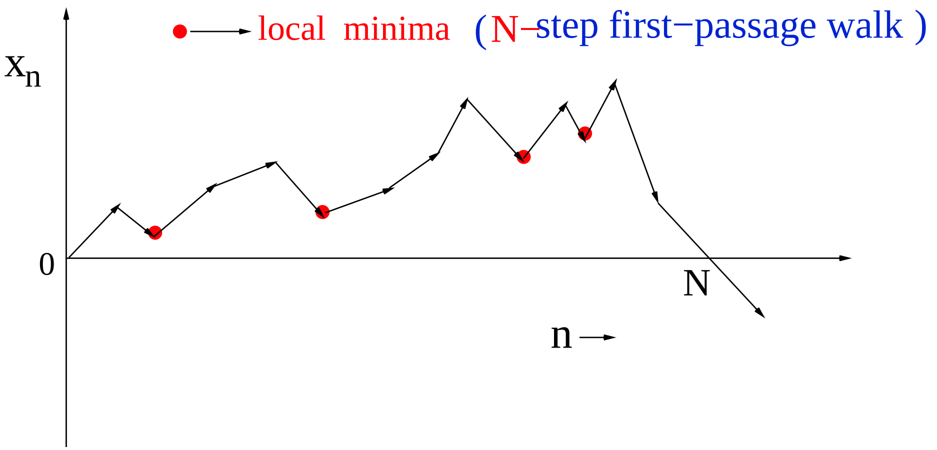

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Remark: $Q^{\rm fp}(0) = \dfrac{3}{4}$

Main results

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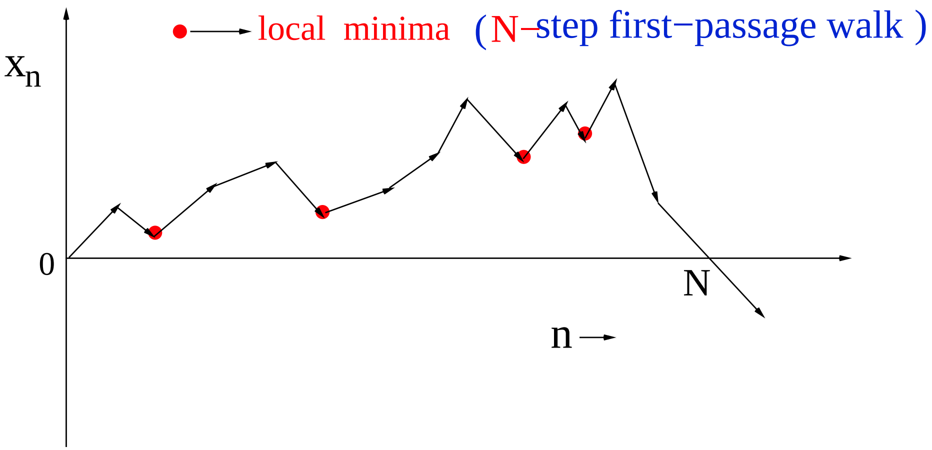

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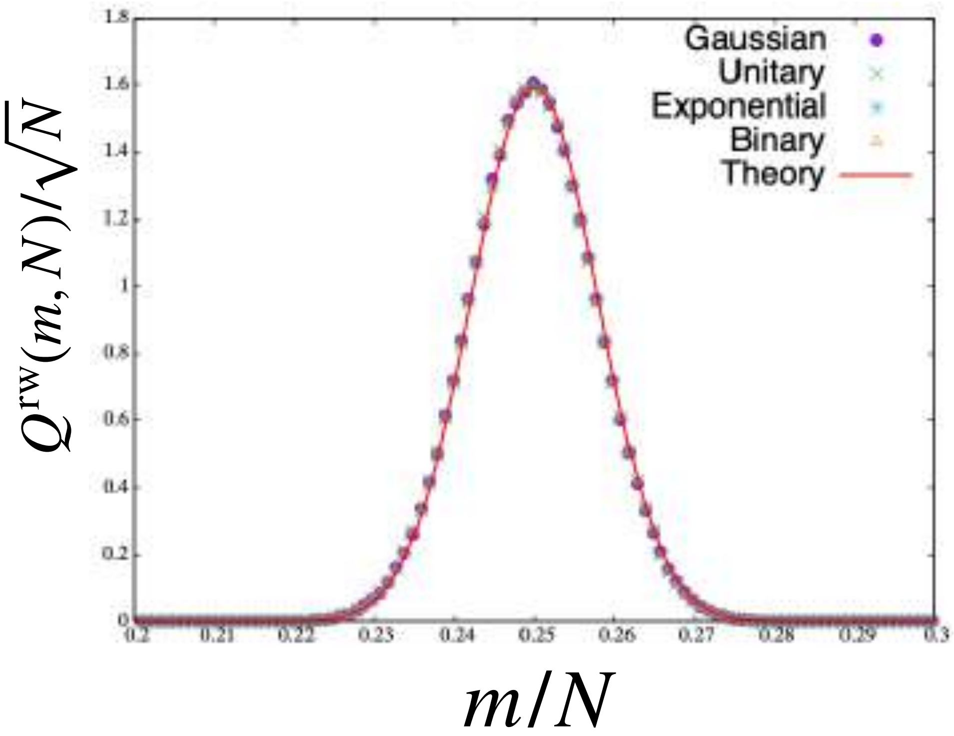

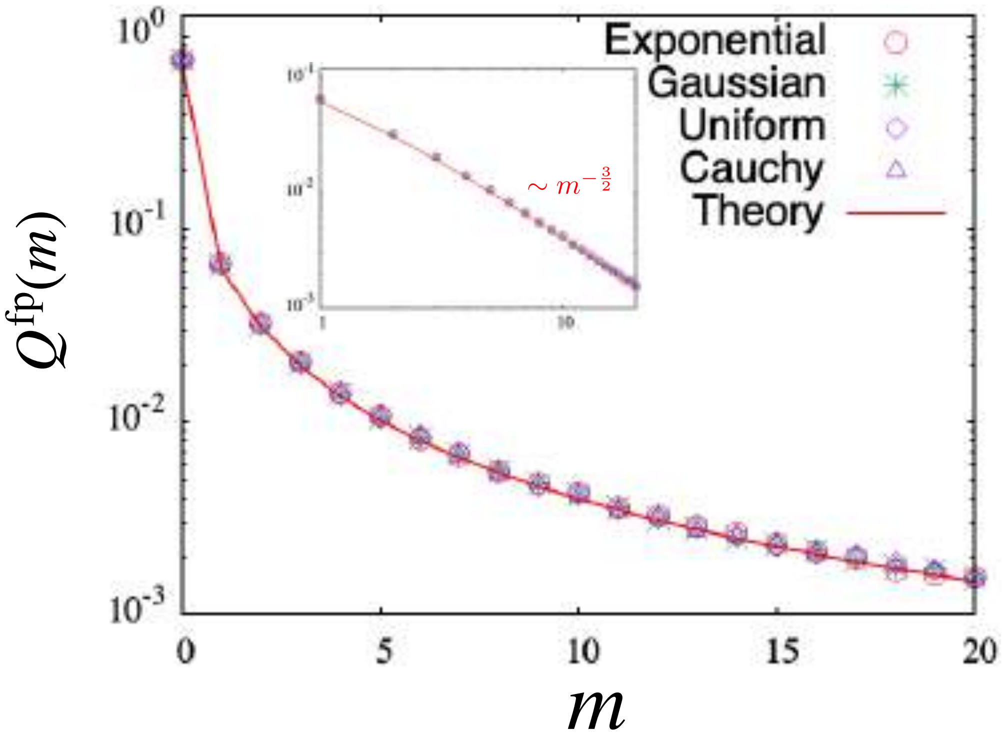

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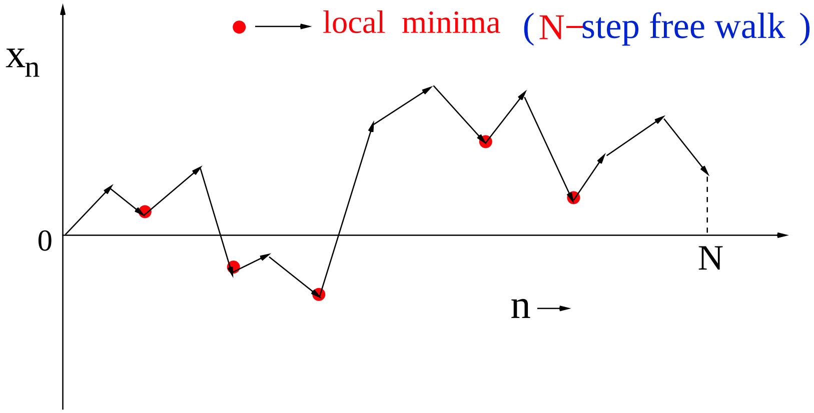

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Main results

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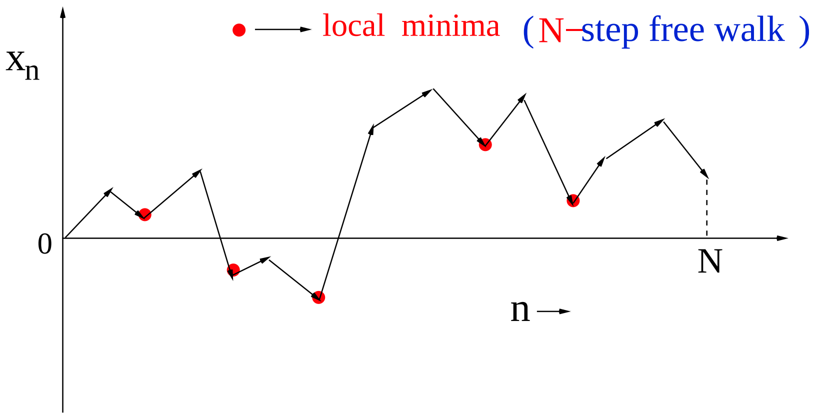

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IID rand. var. distributed with $\phi(\eta)$, continuous and symmetric

- More results for the free random walk
  - Joint probability distribution of the nber of minima $\mathrm{m}_N^{\mathrm{rw}}$ and the nber of maxima $\mathrm{M}_N^{\mathrm{rw}}$ for a free RW of N steps

$$\sum_{N\geq 2} z^N \sum_{m\geq 0} u^m \sum_{M\geq 0} v^M \, \mathrm{Prob.}(\mathrm{m}_N^{\mathrm{rw}} = m, \mathrm{M}_N^{\mathrm{rw}} = M) = z^2 \frac{(2-z) + (u+v) + z\,u\,v}{(2-z)^2 - uvz^2}$$

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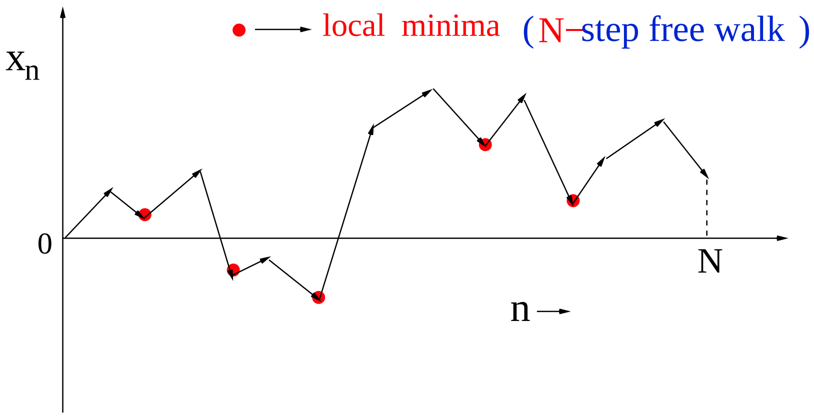

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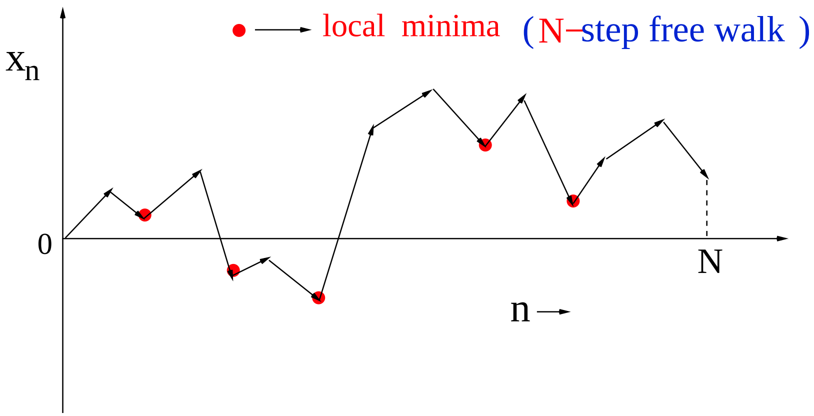
  - Correlation function: $\mathbb{E}(\mathrm{m}_n \mathrm{M}_N) - \mathbb{E}(\mathrm{m}_N)\mathbb{E}(\mathrm{M}_N) = \dfrac{N-3}{16}$

Outline

- Motivations and background
- Main results
- Sketch of the derivation
- Conclusion and perspectives

A 1st approach: recursion relations



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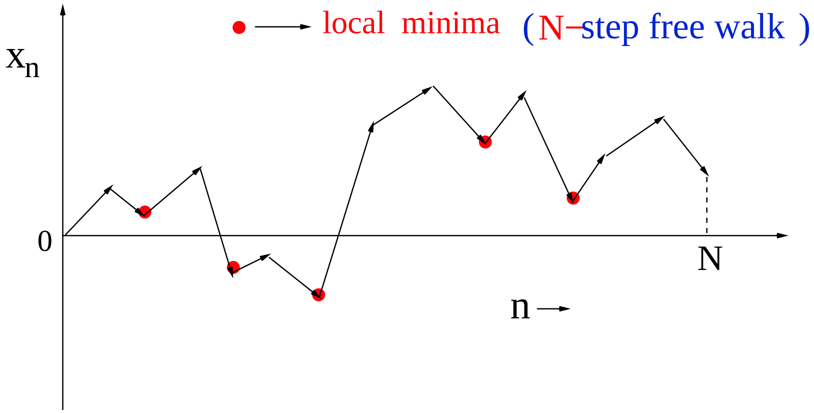

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- Define $Q_{\pm}^{\rm rw}(m,N) = \text{Prob.}\left(\mathrm{m}_N^{\rm rw} = m \quad \& \quad \text{sign}(\eta_1) = \pm\right)$

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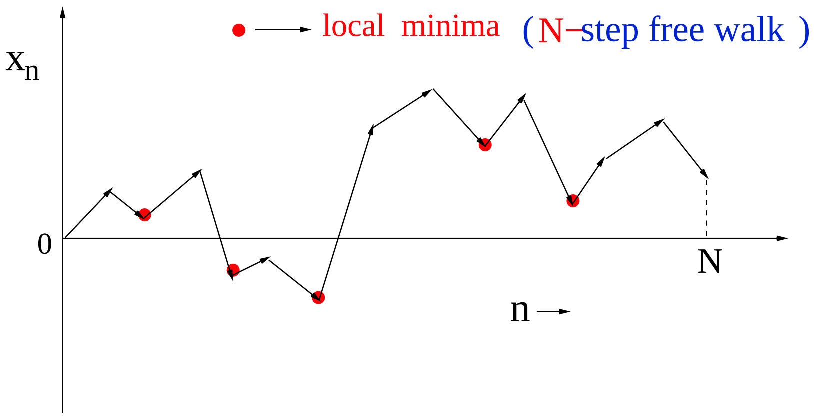

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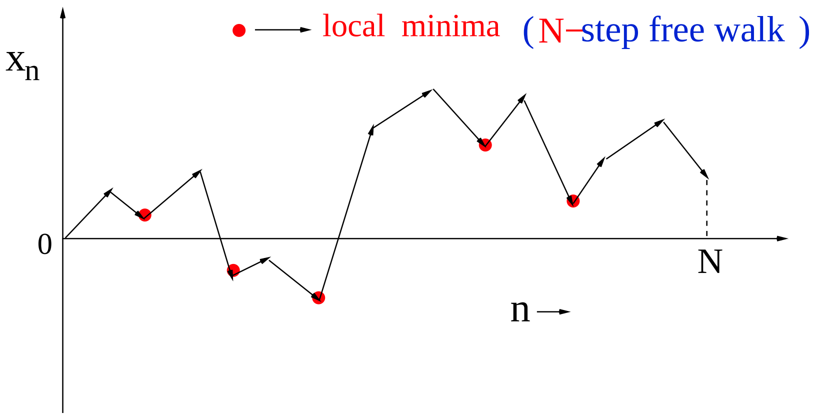

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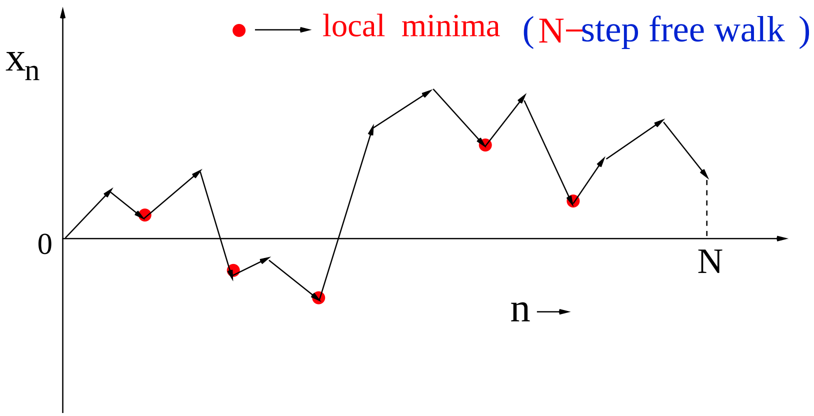

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starting from $Q_+(m,2) = \delta_{m,0}/2 \quad , \quad Q_-(m,2) = (\delta_{m,0} + \delta_{m,1})/4$

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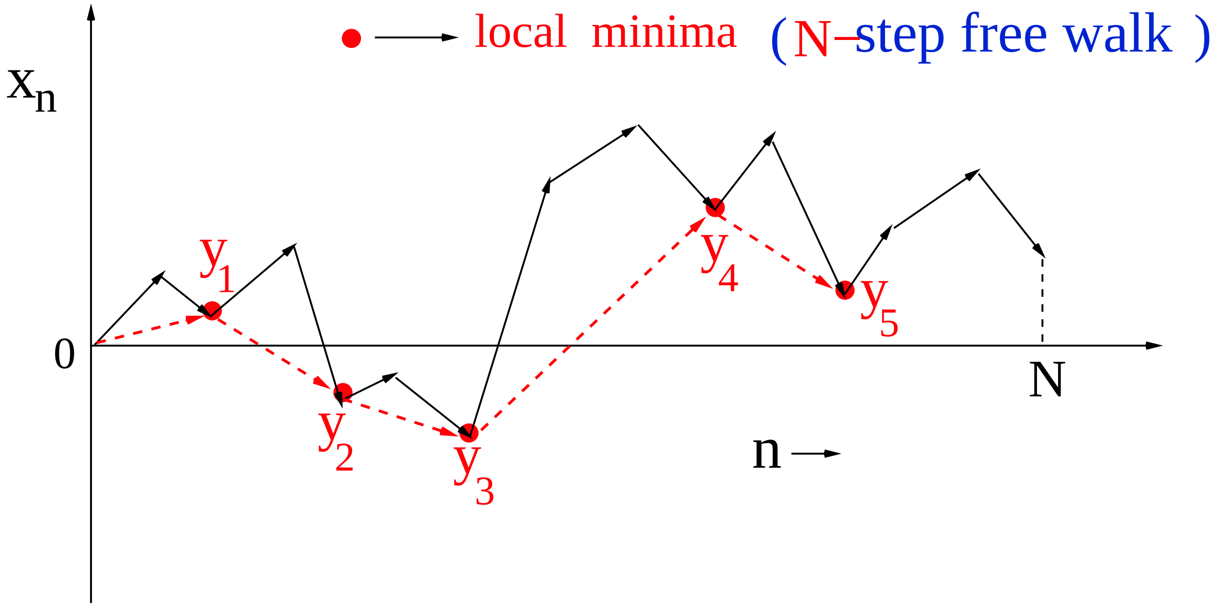
$\Rightarrow$ another method is needed !

A second approach via an auxiliary random walk

- The RW built from the local minima



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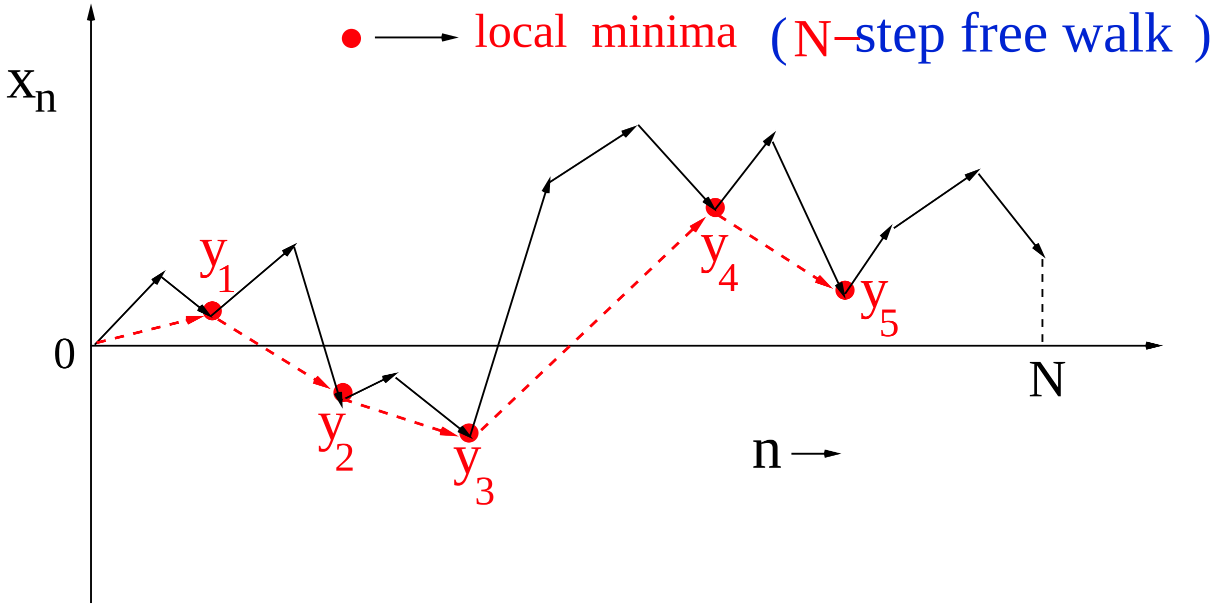

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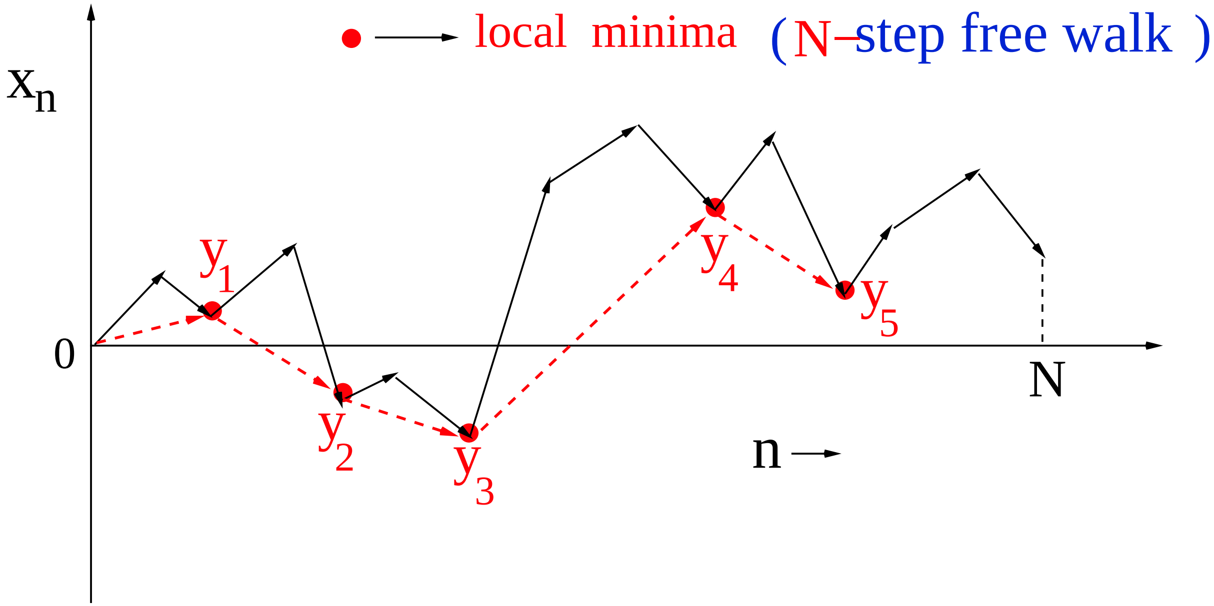

⇨ Note that the number of steps of this effective RW is not fixed

⇨ The effective jump distribution $\psi(y' - y)$ is symmetric and continuous

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Minima of the first-passage random walk

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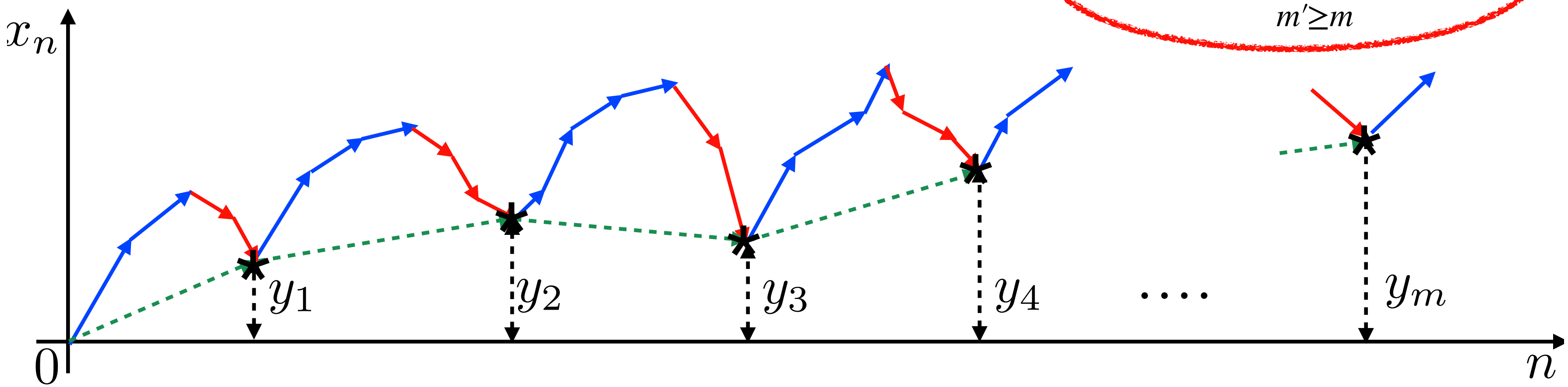

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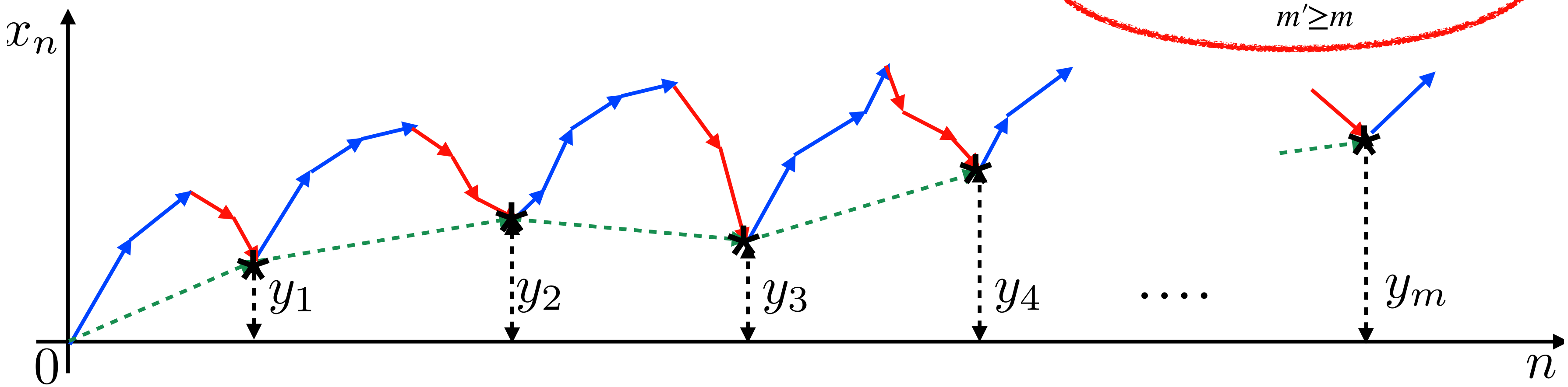

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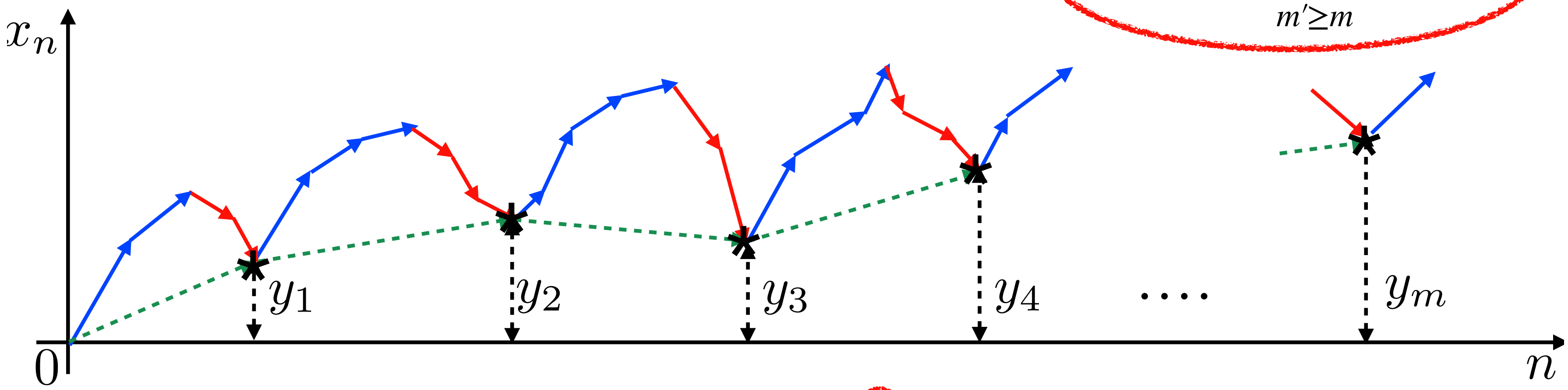

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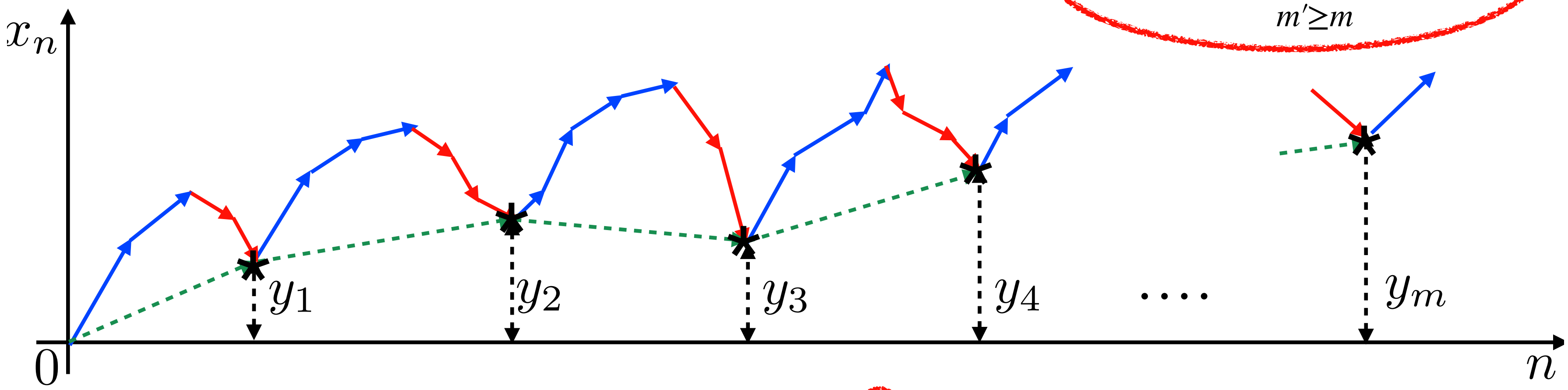

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$$\Rightarrow \quad Q^{\text{fp}}(m) = Q_>^{\text{fp}}(m+1) - Q_>^{\text{fp}}(m) = \frac{1}{2^{2m+2}} \frac{(2m)!}{m!(m+1)!}$$

Outline

- Motivations and background
- Main results
- Sketch of the derivation
- Conclusion and perspectives

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- Are there some natural extensions of these questions in higher dimensions? see, e.g., extensions of the Sparre Andersen theorem in higher dimensions by Z. Kabluchko, V. Vysostsky, D. Zaporozhets

Thank You !